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REVIEW

Re-examining Magnetic Flux Density at the Nanoscale: A Statistical Perspective on Continuum and Discrete Regimes

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Abstract

Magnetic flux density (B) is traditionally interpreted as a continuous field whose flux lines form closed loops, as prescribed by Maxwell's equations. This description is well justified in macroscopic systems, where large ensembles of magnetic dipoles produce statistically smooth fields through spatial averaging. At the nanoscale, however, where only a limited number of dipoles may contribute, the conditions underlying this continuum interpretation become less clear. Here, we reexamine the meaning of magnetic flux density from a classical-statistical perspective, focusing on finite ensembles and isolated magnetic particles. We show that as the number of contributing dipoles decreases, ensemble averaging becomes insufficient to support a statistically stable, coarse-grained field description, even though the underlying electromagnetic fields remain well defined and fully consistent with Maxwell's equations. In this regime, magnetic flux density retains its formal definition, but its interpretation as a robust macroscopic observable becomes strongly dependent on fluctuations and specific dipole configurations. This framework introduces a quantitative criterion based on a critical particle number and provides a consistent description of the transition from ensemble-averaged magnetostatics to discrete dipole behavior. The results clarify the limits of continuum field interpretations at the nanoscale and offer a unified perspective for understanding isolated nanoparticles, small dipole ensembles, and the emergence of classical magnetic behavior from discrete microscopic sources.

Keywords Magnetic flux density · Nanoscale magnetism · Flux quantization · Statistical magnetostatics · Magnetic dipoles · Continuum-to-quantum transition · Isolated nanoparticle · Magnetic field emergence · Single-dipole systems · Quantum magnetism

1 Introduction

Magnetic flux density, B, is a cornerstone of both classical and quantum descriptions of magnetism. In classical electrodynamics, it is defined formally through the magnetic flux Φ_B as the flux per unit area perpendicular to the field direction [1]:

$$B = \frac{d\Phi_B}{dA_\perp}, \text{where} \Phi_B = \int \mathrm{B} \cdot d\mathrm{A}$$

Here, B is a vector field whose orientation specifies the local field direction, and whose magnitude represents the intensity of the field [1]. The conventional interpretation of B is supported by the visualization of magnetic flux lines - continuous curves tangent to $\mathbf{B}$ at every point. Their density per unit area corresponds to the magnitude of the field, thereby serving as a geometric construct that makes the abstract vector field physically intuitive.

1.1 *Historical Development of Flux Density* B

The concept of magnetic flux lines has its roots in Michael Faraday's experimental work in the 1830s and 1840s [2]. Faraday introduced 'lines of force' as a visual and conceptual tool to describe magnetic field geometry. These lines,

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which are defined as mapping the motion of a free north pole, should such an entity exist, under the influence of a magnetic field, were not intended to imply physical discreteness or quantization, but rather to provide an intuitive representation of field direction and relative magnitude whilst noting magnetic field is continuous in free space. Faraday's sketches of curved lines surrounding magnets simply laid the foundation for later field-based descriptions of electromagnetism. James Clerk Maxwell later translated Faraday's intuition into a mathematical framework, where flux lines became visual representations of a continuous divergence-free vector field **B**[3].

In this classical formulation, Maxwell's equations govern the behavior of fields [4]:

$$\nabla \cdot \mathrm{B} = 0 (\text{Gauss's law for magnetism})$$

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t} (\text{Faraday's law of induction})$$

$$\nabla \times \mathrm{B} = \mu_0 \mathrm{J} + \mu_0 \epsilon_0 \frac{\partial \mathrm{E}}{\partial t} (\text{Ampère} - \text{Maxwell law})$$

$$\nabla \cdot \mathrm{E} = \frac{\rho}{\epsilon_0} (\text{Gauss's law for electricity})$$

The condition $\nabla \cdot \mathrm{B} = 0$ implies that magnetic flux lines form closed loops without sources or sinks - a property consistent with the absence of magnetic monopoles in classical physics. For macroscopic systems, where vast ensembles of dipoles create a smooth continuum, this flux-line picture is both intuitive and validated experimentally.

The concept of magnetic flux density has evolved over nearly two centuries, transitioning from a qualitative visualization to a precise mathematical construct in classical physics, and more recently to a quantum-limited quantity whose operational meaning is challenged at the nanoscale. This historical evolution highlights the conceptual tension between classical continuum fields and discrete magnetic entities in modern nanoscale systems.

(a) Faraday's Lines of Force (1830s)

Michael Faraday introduced the notion of "lines of force" to describe the spatial action of magnets and electric currents (Fig. 1a). His sketches depicted curved lines emerging from the north pole of a bar magnet and returning to the south pole. Faraday treated these lines almost as tangible threads, suggesting they were physical carriers of force propagating across space without direct contact. Unlike later mathematical formulations, Faraday's lines were intended to capture both intuitive and physical aspects of magnetic action [2].

(b) Maxwell's Continuum Field (1860s)

James Clerk Maxwell transformed Faraday's ideas into the modern electromagnetic field formalism. He defined the magnetic flux density **B** as a continuous vector field, representable by smooth flux lines whose density per unit area

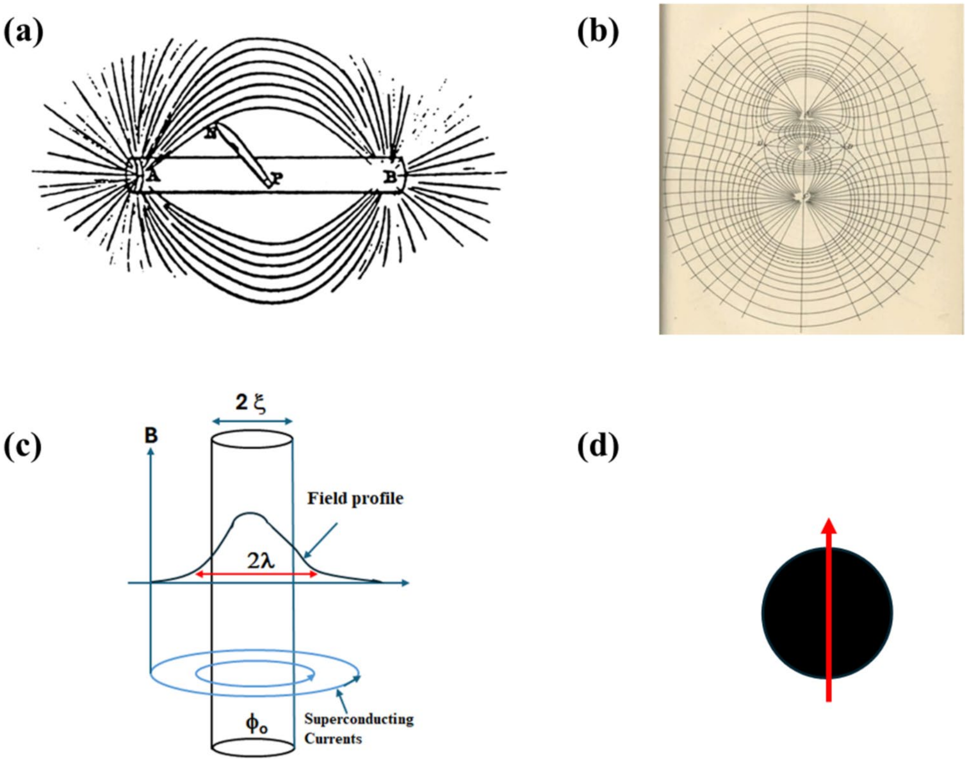

Fig. 1 Evolution of the flux-line concept across physical regimes. (**a**) Faraday's lines of force (1830s) [adapted from ref. 2], depicted as curved trajectories connecting magnetic poles and interpreted as physical "threads" of force. (**b**) Maxwell's continuum field formulation (1860s) [adapted from ref. 3], in which the magnetic field B is represented as a smooth vector field forming closed loops, with flux density proportional to the line density ($\nabla \cdot \mathrm{B} = 0$). (**c**) Quantum fluxoids in superconductors (from 1935 onward) [6], where magnetic flux is quantized in units of $\Phi_0 = h/2e$ and corresponds to discrete, physically real entities [7]. (d) Isolated magnetic nanoparticle [18], illustrating the breakdown of the classical flux-line picture at the nanoscale, where magnetism emerges from discrete magnetic moments rather than a continuous field description

is proportional to magnetic strength (Fig. 1b). These lines were now a visualization of a mathematically rigorous continuum rather than physical threads.

Maxwell formalized their properties with the divergence-free condition:

$$\nabla \cdot \mathbf{B} = 0$$

This condition encodes the absence of magnetic monopoles and ensures that flux lines form closed loops around currents [1, 3, 4]. In bulk systems, magnetic fields arise from both orbital motion of electrons and from their intrinsic spin, with contributions from many electrons averaging statistically to produce smooth, divergence-free fields [5]. In such cases, classical flux lines provide an intuitive and operationally meaningful visualization.

(c) Microscopic Sources of Magnetic Fields

1. Magnetic fields originate from two fundamental microscopic sources:

Magnetic fields arise from the motion of electric charges - whether in microscopic orbital currents or macroscopic current-carrying conductors - as described by the Biot–Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I\, d\mathbf{l} \times (\mathbf{r} - \mathbf{r}^{'})}{|\, \mathbf{r} - \mathbf{r}' \,|^3}$$

where I is the current along an infinitesimal segment $d\mathbf{l}$, $\mathbf{r}$ is the observation point, and $\mathbf{r}^{'}$ is the source point [4].

2. Intrinsic spin angular momentum of electrons, producing a magnetic dipole moment:

$$\mu \approx \mu_B = \frac{eh}{2m_e}$$

where μ_B is the Bohr magneton, e the electron charge, h the reduced Planck constant, and m_e the electron mass [4]. In bulk systems, these contributions average to produce smooth fields; at the nanoscale or single-particle limit, statistical averaging fails, and the classical notion of continuous flux lines becomes less physically applicable.

(3) The Quantum Revolution: Flux Quantization (1935 onward)

The discovery of superconductivity and the rise of quantum mechanics brought a fundamental shift in the understanding of magnetic fields. In 1935, Fritz and Heinz London proposed that magnetic flux in a superconducting loop is quantized in discrete units [6],

$$\Phi_0 = \frac{h}{2e}$$

These flux quanta, or fluxoids, represent physically real, indivisible entities rather than merely visual constructs. Their quantization was later explained microscopically by the BCS theory of superconductivity [7], which attributed the phenomenon to the paired nature of charge carriers. These flux quanta, or fluxoids, are physically real, indivisible objects, rather than mere visual constructs. The quantization of flux was experimentally confirmed by Deaver and Fairbank [5] and independently by Doll and Näbauer [8]. Fluxoids highlight a fundamental conceptual tension: classical physics treats flux lines as infinitely divisible within a continuum, whereas quantum physics enforces discrete, countable units [9, 10].

1.2 From Continuum to Discrete: The Nanoscale Dilemma

The historical development of B from Faraday's physical "lines of force" to Maxwell's continuous field representation concealed an implicit assumption: that the magnetic medium contains an effectively infinite number of microscopic dipoles, such that local fluctuations average out to form a smooth, divergence-free field [4]. At the atomic or nanoscale, however, magnetism originates from discrete sources - orbital currents and intrinsic electron spins [1, 4] - whose individual dipolar fields are highly inhomogeneous and possess localized divergences.

An isolated magnetic dipole generates a well-defined continuous magnetic field that satisfies Maxwell's equations exactly. We do not argue otherwise. Our focus is instead on the operational interpretation of magnetic flux density as a statistically stable continuum observable. In finite ensembles containing only a few dipoles, large temporal and configurational fluctuations can dominate the ensemble average, making the measured field strongly dependent on individual dipole configurations rather than bulk averaging. In this regime, the idea of "flux lines per unit area" becomes a statistical idealization that requires a sufficiently large ensemble to exist for its definition. This issue becomes acute in nanoscale or single-particle magnetic systems, where the number of contributing dipoles N may be too small to justify the continuum limit. The divergence-free property of B remains mathematically exact, but its physical interpretation as a measurable flux density collapses. Thus, while Maxwell's equations hold formally at all scales, the statistical foundation that gives B its physical meaning no longer applies when $N \to 1$ (see Supplemental Information).

In this paper, we re-examine the foundations of magnetic flux density at the nanoscale. We begin by contrasting the continuum-based classical interpretation with the quantum definition of flux quanta. Building on this, we introduce a classical-statistical approach that identifies the limits of the flux-line concept in finite systems. We argue that below a critical size, the number of dipoles is too small to sustain a statistically meaningful flux density, giving rise to a regime where magnetism must be redefined in terms of isolated particles.

The discussion first reviews the conventional classical formulation of flux density and its reliance on ensemble averaging. We then summarize the quantum mechanical perspective, with a focus on flux quantization. Next, a statistical mean-field analysis of finite dipole ensembles is developed, highlighting the critical sizes at which the concept of flux density breaks down (Fig. 1d). Finally, we explore the implications of these findings for nanoscale magnetism, including single-particle systems and magnetic qubits, and conclude with open questions and potential directions for future research.

1.3 Key Issues and Unaddressed Problems

Mathematically, Maxwell's condition $\nabla \cdot \mathrm{B} = 0$ holds identically at all scales, reflecting the absence of magnetic monopoles [3]. In the continuum limit, this equation provides the foundation for visualizing flux lines as closed loops with no sources or sinks. However, the physical interpretation of this condition becomes problematic at the nanoscale. At the level of a single particle - or a particle containing only a few dipoles—there is no smooth, ensemble-averaged flux density. The classical picture of "flux lines" threading a surface loses operational meaning, even though the differential equation itself remains exact. The derivation of macroscopic electromagnetic fields from microscopic sources has been rigorously established in classical electrodynamics through spatial and ensemble averaging of microscopic charges and dipoles, as discussed by de Groot and Suttorp [11], Landau and Lifshitz [12], and related foundational treatments. These works primarily address the thermodynamic limit, where very large ensembles naturally produce smooth macroscopic fields. The present work does not revisit these classical derivations. Instead, it focuses on the opposite limit - finite nanoscale ensembles and isolated magnetic particles - where the number of contributing dipoles may become too small for statistical averaging to produce a stable continuum field. In this regime, we introduce a quantitative framework based on the critical particle number N_c and apply it to isolated Fe_3O_4 nanoparticles to examine the practical lower limits of continuum magnetic flux density.

1.4 Classical vs. Quantum Flux-Line Concepts

In classical physics, flux lines are continuous visual constructs, emerging naturally from ensemble-averaged fields. Their density per unit area scales smoothly with field strength, and their closure reflects $\nabla \cdot \mathrm{B} = 0$. In quantum mechanics, however, flux is discretized: in superconductors, magnetic flux penetrates only in quantized units $\Phi_0 = h/2e$ [6, 7]. These "fluxoids" are physically real and countable, in contrast to the classical visualization of a magnetic field continuum. Crucially, there is no direct analogue of the classical flux-line density picture in the quantum regime. Thus, a conceptual gap exists: how do continuous flux lines in the macroscopic limit emerge from isolated or few discrete dipoles at the nanoscale? This question - bridging classical-statistical and quantum–mechanical descriptions - remains essentially unaddressed in the literature [10].

1.5 Critical Dipole/Particle Size and the Breakdown of Continuum Magnetostatics

The notion of a well-defined magnetic flux density implicitly assumes a statistically large number of contributing dipoles ($N \gg 1$), such that ensemble averaging yields a smooth, divergence-free field in the continuum sense. When the number of dipoles falls below a certain threshold, statistical fluctuations dominate and the field can no longer be treated as continuous. In this discrete regime, the flux density becomes effectively unmeasurable - not because B vanishes, but because the concept of "lines of flux per unit area" loses operational meaning. Theoretical analyses of single-domain nanoparticles and superparamagnetic systems have hinted at such discrete-to-continuum transitions [13, 14], yet the quantitative boundary between these regimes has not been systematically defined, leaving a conceptual gap in magnetostatics.

A useful way to formalize this transition is through a statistical mean-field approach. The emergence of a continuous flux density requires a sufficiently large number of dipoles within a representative volume. For a region of radius r containing dipole density n,

$$N = n\frac{4}{3}\pi r^3 \Rightarrow r \propto (\frac{N}{n})^{1/3}$$

In bulk systems ($N \gg 1$), averaging over many dipoles produces a smooth vector field B that underpins Maxwell's continuum formulation. In this regime, B is defined operationally as the flux per unit area, and its classical interpretation as a continuous, divergence-free field follows from statistical overlap of many microscopic sources (orbital currents and intrinsic spins) [1, 3, 4, 13].

As the system size decreases, N falls as the system size decreases and, once the particle approaches a critical size $\mathrm{r}_{\mathrm{crit}}$ such that $\mathrm{N}_{\mathrm{crit}} \sim 1$, ensemble averaging becomes ineffective and the notion of a smooth flux density loses operational robustness. With only one or a few dipoles contributing, the spatial field retains its full continuity at the level of individual solutions of Maxwell's equations, but the interpretation of 'flux line density' as a statistically stable, coarse-grained descriptor becomes strongly configuration-dependent. Although Maxwell's equations, including $\nabla \cdot \mathbf{B} = 0$, remain exact, their usual macroscopic interpretation in terms of smooth, ensemble-averaged field structures is no longer supported by sufficient statistical averaging in this limit. The isolated-particle regime therefore does not signal a breakdown of electromagnetic theory, but rather the loss of the averaging conditions required for $\mathbf{B}$ to function as a continuum-level observable.

In assemblies of 10 nm Fe_3O_4 nanoparticles, this transition can be interpreted in terms of particle number. When only a few hundred to a thousand particles are present and separated by more than ~ 30 nm, dipole–dipole interactions become negligible, and the particles behave as effectively isolated dipoles. However, a smooth, ensemble-averaged superparamagnetic response - typified by a Langevin-type $M(H)$ curve - requires a statistically large number of dipoles such that the mean magnetization dominates over random fluctuations. For the parameters considered here, the ensemble size required for such averaging is extremely large at weak fields (sub-millitesla), decreases with increasing field strength, and typically remains well above 10^2–10^3 particles for conventional low-field magnetometry.

Roughly speaking, the critical ensemble size N_c may range from $\sim 10^{16}$ at sub-mT fields, to $\sim 10^{12} - 10^{14}$ around tens of mT–0.1 T, and down to $\sim 10^{10}$ only at multi-tesla fields. Small arrays of well-separated nanoparticles thus remain well below this threshold, residing in the discrete-dipole regime where magnetism cannot be described by a continuous flux density.

This isolated-particle model provides a natural bridge between continuum electrodynamics and single-particle magnetism. It reframes the transition not as a failure of Maxwell's equations, but as a loss of the statistical conditions that endow the flux density with its physical meaning. Similar statistical thresholds have been identified in fine-particle magnetism, superparamagnetism, and single-domain limits [13, 14]. The framework introduced here formalizes that boundary, defining a critical particle number N_{crit} below which magnetic flux density ceases to be an operationally measurable quantity.

1.6 *Critical Number of Dipoles, N_{crit}*

Building on the statistical interpretation above, the critical number of dipoles, $\mathrm{N}_{\mathrm{crit}}$, can be defined quantitatively as the minimum ensemble size required for a measurable, ensemble-averaged magnetization. The following estimates illustrate how $\mathrm{N}_{\mathrm{crit}}$ depends on applied field strength for 10 nm Fe_3O_4 nanoparticles at 300 K. The magnetic response of a nanoparticle ensemble depends on the balance between thermal fluctuations and collective averaging.

When the number of magnetic dipoles N in a representative volume is small, random fluctuations dominate, and the notion of a continuous magnetic flux density B breaks down. Only when $N \gg 1$ can statistical averaging produce a smooth, divergence-free field consistent with the continuum picture of magnetostatics.

For N independent dipoles m_i:

$$\langle \mathrm{B} \rangle \propto \sum_{i=1}^{N} \mathrm{m}_i, \delta \mathrm{B} \propto \sqrt{N},$$

so the relative fluctuation scales as

$$\frac{\delta B}{\langle B \rangle} \sim \frac{1}{\sqrt{N}}.$$

To maintain a continuous field, the fluctuations must be small:

$$\frac{\delta B}{\langle B \rangle} \ll 1 \Rightarrow N \gg 1.$$

When $N \lesssim 1$, the concept of a well-defined flux density becomes meaningless - the field exists, but its spatial average fluctuates as strongly as its mean.

$\mathrm{N}_{\mathrm{crit}} \sim 1$ should be interpreted only as a conceptual lower bound marking the disappearance of ensemble averaging. In practical measurements, much larger values ($\mathrm{N} \gg 1$) are required depending on acceptable fluctuation levels and signal-to-noise thresholds.

$$N \geq \frac{1}{\varepsilon^2}.$$

Hence, for 10% relative stability ($\varepsilon = 0.1$), one needs $N \gtrsim 100$; for 1% ($\varepsilon = 0.01$), $N \gtrsim 10^4$. Each nanoparticle has a fixed magnetic moment magnitude m(for 10 nm Fe_3O_4, $\approx 2.5 \times 10^{-19}\ \mathrm{A} \cdot \mathrm{m}^2$). In an applied, external magnetic field of strength H, measured in Am^{-1}, the ensemble-averaged moment per particle is

$$\langle m_z \rangle = m\,L(x), x = \frac{\mu_0 mH}{k_B T},$$

where $L(x)$ is the Langevin function.

For small x(weak fields), $L(x) \approx x/3$.

The standard deviation of the z-projection for one particle is $\sigma \approx m/\sqrt{3}$.

For N independent particles:

$$\text{Mean magnetization: } N\langle m_z \rangle, \text{Fluctuation: } \sqrt{N}\,\sigma$$

The signal-to-noise ratio (SNR) is therefore

$$\text{SNR} = \frac{N\langle m_z \rangle}{\sqrt{N}\sigma} = \sqrt{N}\,\sqrt{3}\,L(x)$$

Requiring SNR ≥ k yields the minimum number of particles:

$$N_c = (\frac{k}{\sqrt{3}\,L(x)})^2$$

For weak fields ($L(x) \approx x/3$):

$$N_c \approx \frac{3k^2}{x^2}, x = \frac{\mu_0 mH}{k_B T}$$

Thus, $N_c \propto 1/H^2$: the weaker the field, the larger the ensemble required for a stable, measurable mean magnetization.

Table 1 lists the parameters used in the numerical examples illustrating the statistical limit of magnetic flux density for 10 nm Fe_3O_4 nanoparticles at 300 K.

Relaxing the SNR requirement to $k = 1$ reduces N_c by $k^2 = 25$.

Physical Interpretation.

- Small ensembles (2–10^3 particles):

Behave as independent, fluctuating dipoles — no smooth M(H) curve.

Table 1 Numerical examples illustrating the statistical limit of magnetic flux density for 10 nm Fe_3O_4 nanoparticles at 300 K. Parameters used are $m = 2.5 \times 10^{-19}$ A · m^2, $k = 5$, and $k_B T = 4.1 \times 10^{-21}$ J . The table lists the applied field H, dimensionless field parameter $x = \mu_0 mH/k_B T$, corresponding Langevin function $L(x) \approx x/3$ in the low-field limit, and the critical number of dipoles N_c required to achieve a signal-to-noise ratio (SNR) of 5

Field H(T)	$x = \mu_0 mH/k_B T$	$L(x) \approx x/3$	N_c (SNR = 5)
1 mT (10^{-3} T)	2.4×10^{-7}	8×10^{-8}	1.3×10^{16}
10 mT (10^{-2} T)	2.4×10^{-6}	8×10^{-7}	1.3×10^{14}
100 mT (10^{-1} T)	2.4×10^{-5}	8×10^{-6}	1.3×10^{12}
1 T	2.4×10^{-4}	8×10^{-5}	1.3×10^{10}

You may observe random telegraph noise or discrete switching events.

- Large ensembles (10^{10}–10^{15} particles):

Statistical averaging dominates; superparamagnetism appears as a continuous Langevin-type response.

- Dependence on H:

At weak applied fields (mT range), N_c is astronomically large; at 0.1 T, N_c is 10^{12}–10^{13};only at multi-tesla fields does N_c approach 10^1–10^3.

- Practical design rule:

Arrays of 100–1000 particles spaced > 30 nm are effectively non-interacting and fall below the superparamagnetic threshold for conventional magnetometry.

To observe a smooth M(H), either increase particle count by several orders of magnitude, raise the applied field, or use a more sensitive detection scheme.

$N_{\text{crit}} \sim 1$ is the conceptual boundary: below it, continuum magnetostatics breaks down.

- N_c(from counting-noise arguments) defines the practical ensemble size needed for a smooth measurable magnetization.
- For 10 nm Fe_3O_4 at 300 K, weak fields (≤mT) require $N_c \gtrsim 10^{14-16}$, while fields near 1 T reduce it to $\approx 10^{10}$.
- Typical "isolated" arrays (100–1000 particles,≥30 nm spacings) remain well below these thresholds, representing the discrete-dipole regime where magnetic flux density is not a continuous quantity.

Superparamagnetism is fundamentally a property of a large ensemble, not an individual particle. A single nanoparticle possesses a magnetic field, but it does not necessarily exhibit the ensemble-averaged response typically associated with macroscopic flux-density measurements. Even small clusters of 100–1000 particles do not exhibit superparamagnetic behavior if their spacing exceeds roughly 30 nm (see Supplemental Information). Only when very large ensembles - on the order of 10^{10}–10^{15} particles - act collectively does the system show the smooth, averaged magnetization characteristic of superparamagnetic behavior in bulk measurements.

1.7 Experimental Validation

Recent advances in experimental techniques offer potential pathways to probe this transition. Nanoscale SQUID magnetometry and scanning probe methods (e.g., spin-polarized

STM, NV-center magnetometry) can detect magnetic signatures at or near the single-spin level [15–17]. Such techniques could provide direct evidence of whether a smooth flux density persists in small ensembles, or whether the concept collapses into discrete dipole moments. Single-domain nanoparticles and magnetic quantum dots already provide indirect evidence that below certain sizes, the classical flux-line picture fails, with magnetic behavior governed instead by spin statistics and quantum confinement effects [16, 17]. However, a systematic experimental program aimed specifically at testing the limits of flux-line applicability has yet to be undertaken.

1.8 Critical Particle Radius

In earlier work, we investigated the physical limits of magnetic flux density in isolated nanoscale particles, with particular attention to the breakdown of classical field concepts as particle size decreases [18]. The central finding was that as the physical dimensions of a magnetic particle shrink, the number of magnetic dipoles available to sustain closed-loop flux diminishes rapidly. This reduction erodes the collective behavior characteristic of bulk systems and amplifies the role of quantum mechanical effects. At such scales, magnetic moments can no longer be treated as continuous sources of a classical vector field but instead manifest as discrete, probabilistic entities governed by superposition and quantization principles.

A key concept introduced was that of the magnetic flux quantum,

$$\Phi_0 = \frac{h}{2e} \approx 2.07 \times 10^{-15}\,\mathrm{Wb},$$

which sets a fundamental scale for magnetic behavior in confined systems [5–10, 18–20]. Because a single Bohr magneton (μ_B) is insufficient to sustain even one flux quantum, the continuity implied by $\nabla \cdot \mathrm{B} = 0$ becomes statistical rather than absolute at the nanoscale. In bulk systems, this divergence-free condition arises from averaging over large numbers of dipoles; in nanoparticles, however, local deviations appear as the continuum approximation collapses. These deviations do not imply the presence of monopoles but reflect the discrete nature of magnetic dipoles.

To quantify the transition, we defined the minimum cross-sectional area required to sustain one flux quantum as

$$A_{\min} = \frac{\Phi_0}{B}$$

with the corresponding critical radius

$$r = \sqrt{\frac{\Phi_0}{\pi B}}$$

For representative field values, we obtained:

$$B = 1\,\mathrm{T} \Rightarrow r \approx 25.7\,\mathrm{nm},$$
$$B = 0.1\,\mathrm{T} \Rightarrow r \approx 81.2\,\mathrm{nm},$$
$$B = 0.01\,\mathrm{T} \Rightarrow r \approx 256.7\,\mathrm{nm}.$$

These results demonstrate an inverse square-root scaling, $r \propto 1/\sqrt{B}$, indicating that stronger internal fields allow smaller particles to sustain a flux quantum, while weaker fields require proportionally larger particles. The critical size thus links particle radius, magnetic field strength, and the validity of classical Maxwellian descriptions.

We also formulated a Statistical Mean-Field Approach in which the magnetic field arises from N discrete dipoles of density n, yielding a characteristic coherence radius

$$r \propto (\frac{N}{n})^{1/3}.$$

For large N, this reproduces continuum behavior and ensures $\nabla \cdot \mathrm{B} = 0$ in an averaged sense. For small N, however, fluctuations dominate, and the field description becomes unreliable. The statistical variance of flux continuity scales as $\mathrm{Var}(\nabla \cdot \mathrm{B}) \propto 1/N$, underscoring the fragility of the divergence-free condition in nanoscale systems.

A direct comparison of the Quantum Flux Quantization model and the Statistical Mean-Field Approach revealed complementary but convergent predictions. While flux quantization establishes a quantum limit based on Φ_0, the statistical model describes the dipole ensemble requirement for maintaining a continuous field. Both predict critical radii in the tens to hundreds of nanometers, identifying a new regime - sub-critical isolated particles - where classical flux lines cease to exist. In this regime, magnetic behavior must be described in terms of discrete spin states and quantized flux, a scenario we termed the quantum magneton regime.

This prior analysis provides the conceptual and mathematical foundation for the present work. By framing the problem in terms of both statistical ensembles and flux quantization, we established two independent but converging constraints on the existence of magnetic flux density at the nanoscale. The current study builds on this framework by critically reexamining the flux-line concept itself and clarifying the conditions under which B retains - or loses - its operational meaning.

1.9 Magnetic Isolation of Fe_3O_4 Nanoparticles

Each Fe_3O_4 nanoparticle is modeled as a uniformly magnetized sphere of diameter.

$D = 10\,\mathrm{nm}$(Radious a = 5 nm). The saturation magnetization was taken as the bulk room-temperature value $M_s \approx 4.8 \times 10^5\ \mathrm{A/m}$, typical for magnetite. If the effective M_s is reduced due to surface disorder or oxidation, the calculated interaction distances will scale down proportionally. The magnetic moment of a single particle is

$$m = M_s V = M_s \frac{4}{3}\pi a^3$$

The dipole–dipole interaction energy, $U_{dd}(r)$, between two magnetic moments aligned along the line joining them (the maximum "head-to-tail" configuration) is

$$U_{dd}(r) = \frac{\mu_0}{4\pi}\frac{2m^2}{r^3}$$

where r is the center-to-center distance and

$$\mu_0 = 4\pi \times 10^{-7}\mathrm{H/m}$$

Thermal fluctuations at temperature T provide a natural comparison energy scale, $k_B T$.

We define "negligible coupling" when $U_{dd} \ll k_B T$, and use three representative thresholds:

$U_{dd} = k_B T$(comparable),

$U_{dd} = 0.1\,k_B T$(small), and.

$U_{dd} = 0.01\,k_B T$(very small).

Particle volume

$$V = \frac{4}{3}\pi\left(5 \times 10^{-9}\mathrm{m}\right)^3 \approx 5.24 \times 10^{-25}\mathrm{m}^3$$

Magnetic moment

$$m = M_s V \approx \left(4.8 \times 10^5\right) \times \left(5.24 \times 10^{-25}\right) \approx 2.51 \times 10^{-19}\mathrm{A}\cdot\mathrm{m}^2$$

Thermal energy (300 K)

$$k_B T \approx 4.14 \times 10^{-21}\ \mathrm{J}$$

Solving $U_{dd}(r) = U_{\text{target}}$ for r:

$$r = \left(\frac{\mu_0}{4\pi}\frac{2m^2}{U_{\text{target}}}\right)^{1/3}$$

Table 2 shows corresponding interaction distances.

Table 2 Estimated dipole–dipole interaction distances for 10 nm Fe_3O_4 nanoparticles at 300 K under different interaction energy conditions. The table lists the dipolar energy U_{dd}, the corresponding separation distance r in meters, and its equivalent in nanometers for the cases $U_{dd} = k_B T$, $0.1\,k_B T$, and $0.01\,k_B T$

Condition	U_{dd}(J)	r(m)	r(nm)
$U_{dd} = k_B T$	4.14×10^{-21}	1.45×10^{-8}	14.5
$U_{dd} = 0.1\,k_B T$	4.14×10^{-22}	3.12×10^{-8}	31.2
$U_{dd} = 0.01\,k_B T$	4.14×10^{-23}	6.73×10^{-8}	67.3

For comparison, when particles are in direct contact r = D = 10 nm:

$$U_{dd} \approx 1.26 \times 10^{-20} J \approx 3\,k_B T$$

indicating strong coupling at contact.

The dipole–dipole interaction energy depends on m^2/r^3, and thus on both the particle moment and separation distance. While U_{dd} itself is independent of temperature (for constant M_s), its relative importance is determined by the ratio $U_{dd}/k_B T$:

- Lower temperatures: $k_B T$ decreases, making $U_{dd}/k_B T$ larger. Dipolar interactions dominate, promoting magnetic alignment, clustering, or glassy behavior.
- Higher temperatures: $k_B T$ increases, thermal agitation overcomes magnetic coupling, and the system approaches superparamagnetic behavior with independent particle moments.

For 10 nm Fe_3O_4 nanoparticles at 300 K, the dipole–dipole interaction energy U_{dd} becomes comparable to the thermal energy $k_B T$ at a center-to-center spacing of approximately 14–15 nm - equivalent to only a few nanometers of surface separation. At this distance, particles remain magnetically coupled, influencing one another's orientation and relaxation dynamics (Fig. 2a). To ensure magnetic isolation ($U_{dd} \ll k_B T$), the spacing must be large enough for thermal fluctuations to dominate over dipolar interactions. This condition is generally satisfied at center-to-center distances of about 30 nm or more (corresponding to a ~ 20 nm surface gap; Fig. 2b). Such separation can be achieved through polymer or surfactant coatings, dispersion in a nonmagnetic matrix, or maintaining low particle concentrations.

Such isolation can be achieved experimentally by:

- coating particles with thick organic or polymer shells,
- dispersing them in a nonmagnetic matrix, or
- maintaining low particle concentrations in solution.

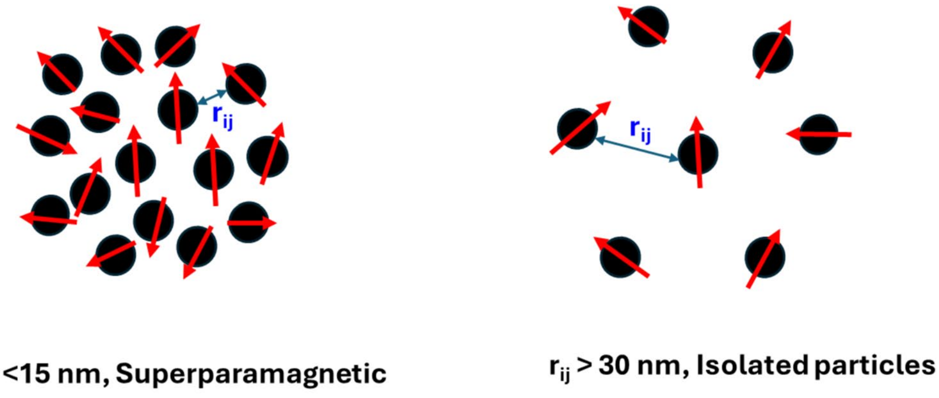

Fig. 2 (a) Fe_3O_4 superparamagnetic nanoparticles with interparticle spacing below 15 nm, where dipole–dipole interactions are significant. (b) Fe_3O_4 nanoparticles with interparticle spacing above 30 nm, considered magnetically isolated with negligible dipolar interactions

Under these conditions, Fe_3O_4 nanoparticles behave as magnetically isolated dipoles, where dipolar correlations are negligible and thermal fluctuations dominate.

1.10 Bridging Classical and Quantum Perspectives

The central challenge in reconciling classical and quantum magnetism lies in the definition of magnetic flux density (B) across scales. In the classical framework, flux density emerges naturally as an ensemble-averaged field, where a large number of dipoles contribute to a statistically smooth continuum [1, 3, 4]. Here, flux lines are a visual abstraction - continuous, divergence-free field curves that represent the operational meaning of $\nabla \cdot \mathrm{B} = 0$. This picture is valid only when the number of microscopic sources is sufficiently large ($N \gg 1$), ensuring statistical coherence of the field.

In the quantum framework, however, magnetic flux is inherently discrete. The smallest unit of magnetic moment is the Bohr magneton $\mu_B = \frac{eh}{2m_e}$, associated with the intrinsic spin of the electron [18, 19]. At the mesoscopic scale, magnetic flux quantization emerges in superconductors, where the fundamental flux quantum is $\Phi_0 = \frac{h}{2e}$, experimentally confirmed in classic experiments by Deaver and Fairbank [5] and Doll and Näbauer [8]. These quanta are not visual abstractions but physically real entities (fluxoids), each corresponding to a discrete, topologically protected state [9, 10].

The key unresolved issue is how these two perspectives connect. The classical flux line represents a smooth, ensemble-averaged construct, whereas the quantum flux quantum represents a discrete and indivisible entity. At the nanoscale, the continuity assumption underlying the classical definition fails with only a single or few dipoles, there is no statistically meaningful flux density. Importantly, this breakdown arises even without invoking quantum mechanics. As shown in the isolated particle model, Below a critical particle size r_{crit}, the number of contributing dipoles falls below $\mathrm{N}_{\text{crit}}{\sim}1$, making the classical flux-line picture physically meaningless, even though $\nabla \cdot \mathbf{B} = 0$ remains mathematically exact..

This observation reframes the debate. The failure of the classical flux concept at the nanoscale is not solely a consequence of quantum mechanics - it is first a statistical problem. Quantum mechanics then sets additional discrete limits: the Bohr magneton for single spins [18, 19], and flux quanta in superconductors [5, 9, 10]. Thus, the transition from single dipoles to macroscopic systems involves a two-step emergence:

1. Statistical emergence of a smooth field from ensembles of dipoles (classical continuum limit).
2. Quantum constraints that discretize magnetic moments and flux in systems governed by coherence effects.

This dual perspective establishes a framework for studying the emergence of classical magnetostatics from discrete magnetic entities. It also highlights the open question: *is there a continuous bridge between flux quanta and flux lines, or does the transition remain fundamentally discontinuous?* Addressing this question is critical for understanding the magnetic behavior of ultra-small systems, from single-domain nanoparticles [13, 14, 16, 17] to quantum bits based on isolated spins or nanoclusters [15, 21, 22].

2 Concluding Remarks

The present study reexamines the physical interpretation of magnetic flux density at the nanoscale by combining a statistical mean-field perspective with insights from quantum flux quantization. A central result is that the classical notion of magnetic flux lines is not an intrinsic property of individual dipoles but a statistical construct emerging only from large ensembles of magnetic moments. In this view, B is not a fundamental microscopic observable - it is a coarse-grained descriptor of flux density resulting from the collective behavior of many dipoles, consistent with the continuum formulations of Landau and Lifshitz [12] and the pedagogical treatments in Reitz et al. [23] and Purcell and Morin [24].

Our results identify a characteristic particle size below which the classical field description becomes inapplicable. When the number of contributing dipoles falls below a critical level, the smooth, divergence-free field - though mathematically exact - ceases to be physically realizable. This collapse is not a violation of Maxwell's laws but a breakdown of their statistical foundation: isolated systems lack sufficient averaging to sustain the continuum limit assumed in classical magnetostatics [25]. The divergence condition $\nabla \cdot B = 0$ remains valid in the operator sense of quantum electrodynamics (QED), yet its classical meaning evaporates at the scale of a single magnetic particle.

Importantly, this conclusion arises without invoking explicitly quantum effects such as flux quantization or spin superposition. The disappearance of B as a measurable field below a certain scale follows entirely from classical–statistical reasoning, aligning with the principles of macroscopic electrodynamics in Landau & Lifshitz [12] and the microscopic spin framework described by Poole [26]. This perspective sidesteps the conventional ambiguities of the quantum–classical boundary while illuminating an overlooked regime in single-particle magnetostatics, where the field must be treated as a sum of discrete dipolar contributions rather than a continuous vector field.

When combined with prior work on flux quantization, the present analysis reveals two complementary limits governing nanoscale magnetism:

(i) a quantum limit, defined by the magnetic flux quantum Φ_0, setting the smallest possible field–area product for flux continuity; and
(ii) a statistical limit, determined by the number of dipoles N, which dictates how much averaging is required for B to acquire physical meaning.

Together, these limits suggest that once the characteristic size falls to tens of nanometers, the operational concept of magnetic flux density ceases to hold - a point consistent with Néel's early treatment of fine-particle magnetism [25] and more recent continuum analyses [27].

Looking ahead, several paths merit exploration. On the theoretical side, extending the mean-field model to incorporate dipole–dipole coupling, anisotropy, and surface effects could refine estimates of the critical size and connect more directly to experimental systems. On the experimental side, advanced techniques such as nitrogen-vacancy magnetometry [21], nanoscale SQUID sensing [21], and spin-polarized STM [15] now offer the spatial resolution required to probe this discrete-to-continuum crossover directly. Observing how isolated dipoles transition toward ensemble-averaged magnetization would provide a decisive test of the statistical framework proposed here.

Ultimately, this work reframes nanoscale magnetostatics as a problem of emergence. Classical magnetic fields and the familiar image of continuous flux lines are not fundamental entities; they materialize only when a sufficient number of dipoles cooperate statistically. Below that threshold, magnetism must instead be described through discrete magnetic moments, stochastic dipolar fluctuations, or flux quantization phenomena. Recognizing where this transition occurs clarifies the physical meaning of $\nabla \cdot B = 0$ at the nanoscale and opens new directions for single-dipole magnetism, with potential implications for spin-based quantum technologies, magnetic qubits, and the design of nanoscale magnetic materials.

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Author Contribution D. S. conceived and developed the original concept of the study, formulated the theoretical framework, and identified the key physical questions addressed in this work. D. S. derived the analytical models, performed all mathematical calculations and scaling analyses, and carried out the statistical estimates presented in the manuscript. D. S. wrote the main manuscript text and prepared all figures (Figs. 1–2), tables, and conceptual schematics. D. S. also secured NSF funding that supported this research. D. C. contributed to scientific discussions, provided critical feedback on the theoretical interpretation, and edited the manuscript for clarity and organization. All authors reviewed the manuscript, discussed the results, and approved the final version.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval and Consent to Participate This study did not involve human participants, animals, or clinical data; therefore, ethics approval was not required.

Conflicts of interest The authors declare no competing interests.

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