MRS Communications
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https://doi.org/10.1557/s43579-025-00818-6

MRS Communications
Research Letter

Critical size for sustaining magnetic flux: Revisiting $\nabla \cdot B = 0$ at the nanoscale

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(Received 5 June 2025; accepted 27 August 2025)

Abstract

We establish the theoretical construct of an isolated magnetic nanoparticle, defined as a single magnetic particle free from inter-particle dipolar interactions. Within this framework, we demonstrate that the classical Maxwellian condition $\nabla \cdot B = 0$ breaks down at a critical nanoscale size (~ 25.7 nm), below which the condition loses physical relevance for isolated particles. At these dimensions, discrete dipole effects, flux quantization, and incomplete flux closure dominate, replacing the smooth field behavior of bulk systems. The requirement of a minimum cross-sectional area to sustain a single magnetic flux quantum provides a natural size threshold for magnetic stability at the nanoscale. This work establishes a conceptual and quantitative boundary between classical and quantum magnetism, offering new insights into nanoscale magnetic phenomena and guiding the design of novel magnetic nanostructures, spintronic elements, and quantum magnetic devices.

Introduction

Maxwell's equations form the foundation of classical electrodynamics, describing how electric and magnetic fields arise from charges and currents.[1–3] Among these, the equation

$$\nabla \cdot B = 0 \tag{1}$$

is particularly profound. It states that the divergence of the magnetic field B is zero at all points in space, implying that magnetic field lines have no beginning or end. Physically, this means that magnetic monopoles—isolated north or south magnetic poles—do not exist within the classical framework. Every magnetic source must be a dipole, producing closed-loop magnetic flux lines that emerge from a north pole and return to a south pole, maintaining flux continuity throughout space.

This condition, however, is rooted in the continuum approximation of matter, which assumes a sufficiently large ensemble of magnetic moments. In macroscopic systems—where countless aligned atomic dipoles collectively generate a coherent magnetic field—the divergence-free condition $\nabla \cdot = 0$ holds reliably. However, as the size of an isolated magnetic particle is progressively reduced, the number of contributing dipoles decreases, leading to a measurable decline in the magnetic flux density (B). At a certain critical scale—where only a few, or even a single, magnetic moment remains—the assumptions underlying the classical field description begin to fail. Consequently, the applicability of $\nabla \cdot B = 0$ becomes questionable, not due to a fundamental violation, but because the statistical basis for the field's continuity no longer holds.

To address this regime, a new conceptual framework is required: *an ideal magnetic particle*. Analogous to the ideal gas in thermodynamics, this ideal particle is defined by the absence of magnetic interactions with other particles. This distinction is crucial, as many-particle systems in the ~ 10 nm range often exhibit superparamagnetic behavior due to inter-particle dipole interactions. While the critical size of an ideal particle may also lie within the nanometer scale, its magnetic properties are governed solely by internal dipoles, free from external magnetic coupling.

In this study, we examine the limits of classical electrodynamics in describing the behavior of an ideal magnetic particle, particularly as its size approaches the regime where the assumptions of a continuous medium no longer hold. Our central objective is to identify the critical size below which the classical framework—specifically the divergence-free condition $\nabla \cdot B = 0$—ceases to provide an accurate or physically meaningful description of magnetic behavior. As the particle size decreases, the finite number of magnetic moments and the growing influence of quantum effects, such as spin quantization and magnetic flux discreteness, undermine the validity of the continuum approximation. This work aims to quantify that threshold and demonstrate the breakdown of classical field theory at the nanoscale.

Our findings introduce a new physical perspective that accounts for the quantum mechanical nature of magnetism at the nanoscale. In this regime, models must move beyond averaged field descriptions to incorporate the discrete and statistical behavior of individual spins, as well as the incomplete flux closure characteristic of quantum-confined systems. Developing such a framework is essential for accurately describing the magnetic properties of nanoparticles, single-spin systems, and emerging quantum magnetic materials—domains where classical laws break down and quantum effects dominate.

Determining critical size via quantum flux quantization

As the physical dimensions of an isolated magnetic particle shrink, the number of magnetic dipoles capable of sustaining a closed-loop magnetic flux diminishes. Concurrently, the system begins to lose its bulk characteristics, and quantum mechanical effects become dominant. At such scales, magnetic moments no longer behave as continuous classical field sources but as discrete, probabilistic entities governed by quantum principles such as superposition. The concept of a magnetic flux quantum:[4–7]

$$\Phi_0 = \frac{h}{2e} \approx 2.0678 \times 10^{-15} Wb \tag{2}$$

where h is Planck's constant and e is the elementary charge.

The generation of even a single quantum of magnetic flux may exceed what a lone magnetic dipole (e.g., a single Bohr magneton) can sustain. Under these conditions, the classical notion of continuous magnetic field lines may appear to break down—not because Maxwell's equation ∇·B = 0 is violated, but because its applicability becomes statistical rather than absolute. In bulk systems, ∇·B = 0 emerges from the collective behavior of many dipoles producing smooth, continuous fields. Below a certain critical size, this collective behavior collapses, and the classical field description no longer accurately captures the magnetic properties. As a result, localized deviations from flux continuity may arise—not due to the presence of true magnetic monopoles, but due to the discrete quantum nature of magnetic moments dominating the system and invalidating the continuum approximation.

To quantify this critical size, we calculate the minimum cross-sectional area required to sustain a single quantum of magnetic flux. Assuming the particle supports a uniform internal magnetic flux density **B**, this area is::

$$A_{\min} = \frac{\Phi_0}{B} \tag{3}$$

From this, the critical radius is calculated assuming a circular cross-section:

$$r = \sqrt{\frac{A_{\min}}{\pi}} = \sqrt{\frac{\Phi_0}{\pi B}} \tag{4}$$

We apply this to three representative values of **B**:

1. High internal field (B = 1 T):

$$A_{\min} = \frac{\Phi_0}{B} = \frac{2.07 \times 10^{-15} \sim Wb}{1 \sim T}$$
$$= 2.07 \times 10^{-15} m^2 \Rightarrow r \approx 25.7 nm$$

2. Moderate field (B = 0.1 T):

$$A_{\min} = \frac{\Phi_0}{B} = \frac{2.07 \times 10^{-15} \sim Wb}{0.1 \sim T}$$
$$= 2.07 \times 10^{-14} m^2 \Rightarrow r \approx 81.2 nm$$

3. Low field (B = 0.01 T):

$$A_{\min} = \frac{\Phi_0}{B} = \frac{2.07 \times 10^{-15} \sim Wb}{0.01 \sim T}$$
$$= 2.07 \times 10^{-13} m^2 \Rightarrow r \approx 256.7 nm$$

These calculations illustrate that the critical radius below which a particle cannot sustain more than one quantum of magnetic flux increases as the internal field decreases. This relationship highlights a fundamental link between particle size, flux quantization, and the validity of classical magnetic field laws.

Figure 1 shows the flux density dependence of the critical radius. As shown in this figure, As B increases, the required area and radius decrease. This means that stronger internal magnetic fields allow smaller particles to sustain a single flux quantum. Conversely, weaker fields require larger particles to do the same. As expected, the critical radius increases as the magnetic flux density decreases, following the inverse square root relationship:

$$r \propto \frac{1}{\sqrt{B}} \tag{5}$$

Statistical mean-field approach

Let us model the magnetic field as arising from **N** discrete magnetic dipoles μ_i, located at positions r_i.[8,9] Each dipole contributes to a magnetic field:

$$B_i(r) = \frac{\mu_0}{4\pi}\left[\frac{3(\mu_i \cdot \hat{r}_i)\hat{r}_i - \mu_i}{|r - r_i|^3}\right] \tag{6}$$

The total magnetic field is the sum over all dipoles:

$$B(r) = \sum_{i=1}^{N} B_i(r) \tag{7}$$

For large N, this summation approximates an integral over a continuous magnetization field, and the condition ∇·**B** = 0 holds approximately in the mean-field sense. However, for small N, statistical fluctuations and the discrete nature of sources lead to local deviations from the divergence-free condition, resulting in a nonzero divergence in a statistical or spatially averaged sense.

In the present analysis, we have neglected explicit dipole–dipole interactions, treating each magnetic moment as an independent contributor to the total field. This simplification is consistent with the ideal magnetic particle model, which considers an isolated nanoscale system whose magnetic properties arise solely from its internal dipoles. In real particles, dipole–dipole interactions can influence local field distributions and contribute to collective behavior. However, for particles at or below the critical size, the number of dipoles is small, and collective interactions are minimal, so the statistical argument for the breakdown of the continuous-field description remains robust. Incorporating

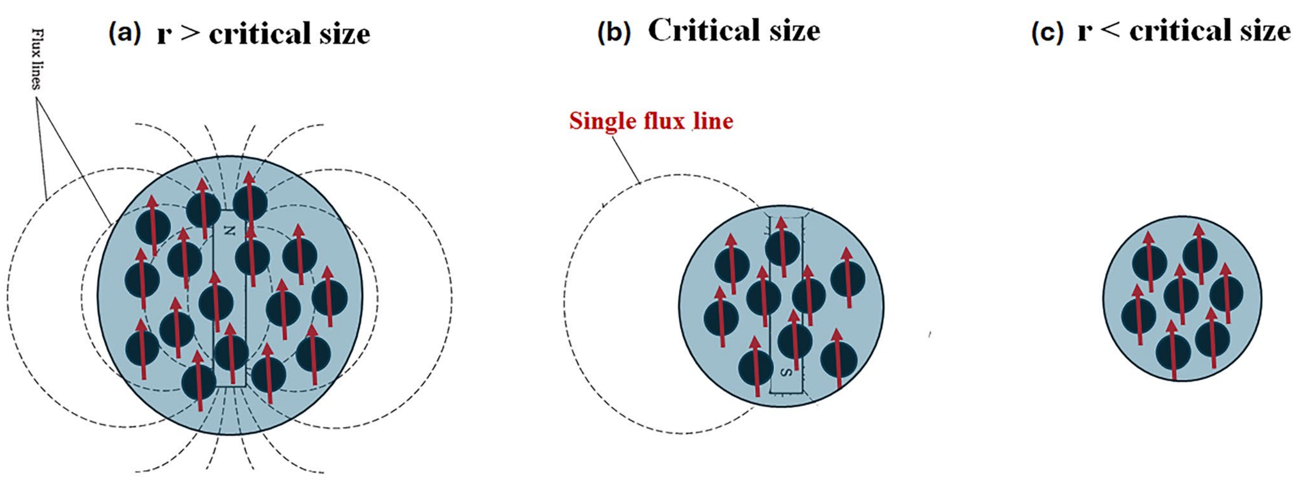

Figure 1. Schematic illustration of size-dependent magnetic flux behavior in an isolated single particle. (a) A relatively large particle with fully aligned internal dipoles generates multiple observable magnetic flux lines, consistent with classical magnetic field continuity. (b) At the critical size, the particle supports only a single flux line, with a limited number of dipoles insufficient to sustain full flux closure. (c) A sub-critical particle, though containing similarly aligned dipoles, exhibits no observable flux lines due to intrinsic size effects that suppress classical magnetic field manifestation below the critical threshold. The particle is considered "ideal," meaning it is magnetically isolated and interacts only through its internal dipoles.

dipole–dipole couplings would refine local field estimates but would not qualitatively alter the primary conclusion that below the critical size, the continuum approximation and the classical interpretation of $\nabla \cdot B = 0$ become physically unreliable.

We can quantify this using the statistical variance:

$$Var(\nabla \cdot B) \propto \frac{1}{N} \tag{8}$$

Assume the system occupies a spherical volume:

$$V = \frac{4}{3}\pi r^3 \tag{9}$$

which translates into a critical volume:

$$V_{critical} = \frac{N_{critical}}{n} \Rightarrow r_{critical} = \left(\frac{3}{4\pi n}\right)^{1/3} \tag{10}$$

which corresponds to the radius of a sphere that contains, on average, one dipole, assuming a uniform number density n.

Using a typical atomic density of magnetic materials:

$$n \sim 10^{23} m^{-3}$$

This gives the radius of a sphere that contains exactly one dipole on average, i.e., assuming a single dipole per unit volume at a number density n. So, it's independent of N—hence the constant result (~13.4 nm for $n = 10^{23}$ m^{-3}). For an actual volume that contains N dipoles, the correct formula is:

$$r = \left(\frac{3N}{4\pi n}\right)^{1/3} \tag{11}$$

where $n = 10^{23}$ m^{-3} is the atomic density of magnetic dipoles. Table I shows the critical radius values at different numbers of dipoles.

Currently, direct experimental measurements of the critical radius for sustaining a single magnetic flux quantum in isolated nanoparticles are scarce, largely due to the difficulties in probing individual spins at the nanoscale. Indirect support can be found in studies of single-domain nanoparticles and flux quantization phenomena in superconducting systems, which suggest that discrete flux effects become relevant at nanometer scales. Future experimental work, employing high-resolution techniques such as spin-polarized scanning tunneling microscopy or nanoscale SQUID magnetometry, could provide quantitative verification of the critical radii predicted in Table I. These studies would allow direct observation of the transition from classical continuum behavior to the quantum-discrete regime of magnetism described in this work.

The critical radius scales with the cube root of the number of dipoles:

$$\frac{N}{n} = \frac{4}{3}\pi r^3 \Rightarrow r \propto \left(\frac{N}{n}\right)^{1/3} \tag{12}$$

This analysis indicates how larger systems—those comprising more dipoles—require proportionally larger volumes to preserve the same statistical coherence of the magnetic field.

Table I. Critical radius as a function of number of dipoles N.

Number of dipoles N	Critical radius
1	13.4 nm
10	28.8 nm
100	62.0 nm
1000	134 nm
10,000	288 nm

As shown in Fig. 2, the blue curve represents the critical radius predicted by the Quantum Flux Quantization model, which decreases with increasing magnetic flux density B, consistent with an inverse square root dependence. By contrast, the red curve, based on the Statistical Mean-Field Approach, shows a critical radius that increases with the number of dipoles N, following a cube-root scaling law.

Despite emerging from fundamentally different physical principles, both models predict similar critical sizes—from tens to hundreds of nanometers—under realistic parameter ranges. This convergence reflects two complementary but independent constraints on the existence of magnetic flux at the nanoscale. The flux quantization model sets a quantum limit for the minimum field strength needed to sustain a single magnetic flux quantum in a finite volume, whereas the mean-field model describes the ensemble requirement needed to maintain a statistically divergence-free field through collective dipole averaging.

Importantly, this framework allows us to identify a previously unexamined regime: that of isolated magnetic particles below the critical size, where neither model can ensure the presence of flux lines. In this regime, the magnetic field may become physically undefined or require a new quantum–mechanical description, since no classical field structure can be sustained. This distinction challenges the traditional assumption that even nanoparticles behave collectively and opens the door to single-dipole magnetostatics, or what we refer to here as a quantum magneton regime. These findings lay a foundation for future investigations into magnetic confinement, non-interacting spin systems, and ultra-small magnetic qubits where classical field assumptions no longer hold.

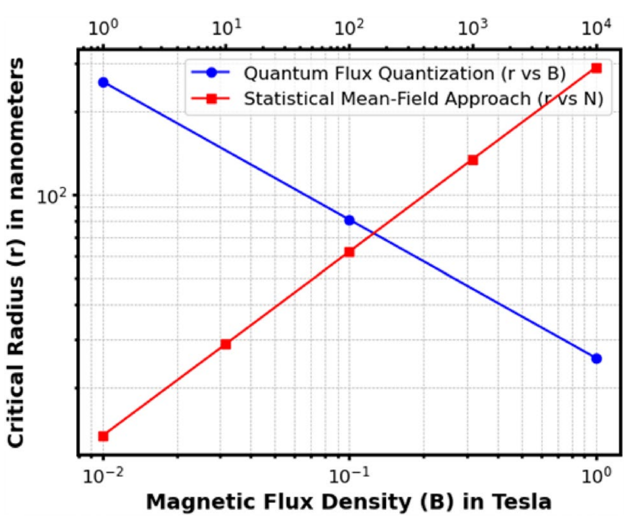

Figure 2. comparison of critical radius calculated from Quantum Flux Quantization and Statistical Mean-Field Approach.

Implications for $\nabla \cdot B = 0$ in nanoscale and quantum–classical transition in magnetism

Below the calculated threshold radius (e.g., ~25.7 nm for B = 1 T), magnetic systems can no longer sustain sufficient flux to ensure loop closure. As a result, the classical condition $\nabla \cdot B = 0$ loses physical applicability. In this regime, magnetic fields are no longer continuous, differentiable entities, but instead manifest as discrete flux quanta tied to localized spin states. This apparent violation of Maxwell's law does not imply a flaw in the theory itself, but rather reflects the breakdown of continuum magnetostatics from a statistical and quantum mechanical standpoint. At such scales, magnetic behavior arises from the probabilistic configuration of individual spin states, necessitating new formulations rooted in quantum electrodynamics (QED).

In deriving the critical radius in Eqs. (3) and (4), we assumed a uniform internal magnetic flux density B within the particle. This approximation is intended as a first-order, idealized treatment, consistent with the "ideal magnetic particle" model proposed in this work. In real nanoscale systems, the internal field distribution will generally be nonuniform due to surface effects, atomic lattice anisotropy, and quantum confinement of spin states. However, when the particle size is sufficiently small and the magnetic moments are strongly exchange-coupled, the internal magnetization can be treated as nearly uniform, making the assumption reasonable for estimating the order of magnitude of the critical radius. More sophisticated treatments could relax this assumption by incorporating spatially varying fields and quantum spin textures, but such refinements would not qualitatively alter the main conclusion—that below the critical size, the continuum-field description loses its physical applicability.

The equation $\nabla \cdot B = 0$ represents a foundational principle of classical electromagnetism: the nonexistence of magnetic monopoles within a continuous magnetic field. This condition is rigorously valid in the macroscopic limit, where B can be treated as a smooth vector field due to statistical averaging over vast numbers of atomic dipoles.

However, as we scale down to *isolated* nanoscale magnetic particles—especially those whose dimensions approach the magnetic flux quantum scale (~25.7 nm for B = 1 T)—the continuum assumption collapses. At these dimensions, smooth classical fields are replaced by discrete magnetic moments, spin states, and quantized flux tubes (Fig. 1). The magnetic field is no longer a well-defined spatial function, but an emergent phenomenon of quantum mechanical interactions.

To formalize this transition, we introduce the concept of an **ideal magnetic particle**: an isolated nanoscale system whose magnetic properties arise solely from a finite number of internal dipoles, free from ensemble effects or external interactions (unlike superparamagnetic systems).[10–16] This model serves as a magnetic analog to the ideal gas molecule in thermodynamics. As particle size shrinks below the critical threshold, the classical field-based view becomes inadequate. In this regime, $\nabla \cdot B = 0$ ceases to hold strictly, and statistical fluctuations dominate due to the discrete nature of magnetic sources. This shift marks the onset of quantum behavior and challenges the utility of traditional field-line-based descriptions.

Surface effects and shape-dependent demagnetizing fields are well known to influence the internal magnetization of nanoscale particles, especially for non-spherical geometries such as ellipsoids. In the present study, we focus on idealized, magnetically isolated particles with strong internal alignment of dipoles. In this regime, the number of magnetic moments is sufficiently small that surface-induced demagnetizing fields modify the internal flux distribution but do not fundamentally alter the statistical arguments regarding the breakdown of the continuous-field approximation. Consequently, while real particles may exhibit nonuniform magnetization due to boundaries, the key conclusion remains robust: below the critical size, the classical continuum interpretation of $\nabla \cdot B = 0$ becomes physically unreliable, and quantum-discrete effects dominate.

In QED, the magnetic field operator still satisfies $\nabla \cdot B = 0$ mathematically, but this applies to operator algebra and expectation values—not to classical vector fields in real space. In confined, quantum-scale systems, classical field lines do not exist; instead, magnetic behavior arises through quantized flux and spin-orbital dynamics. This situation parallels the divergence of the electric field, $\nabla \cdot E = \rho/\varepsilon_0$. Just as classical charge density ρ loses meaning at the single-electron scale—being replaced by probability amplitudes—so too must magnetism be reconceived in terms of quantum states.

The breakdown of Maxwell's equations at the nanoscale is not merely abstract. As A. Zangwill emphasizes in *Modern Electrodynamics*,[17,18] Maxwell's framework is ideally suited for bulk materials, where spatial averaging renders fields smooth and continuous. When applied to systems with only a handful of spins, however, its descriptive power wanes. Building on this insight, we identify a concrete threshold (~25.7 nm for 1 T), below which classical electrodynamics becomes an approximation rather than a governing principle. This boundary offers a physically meaningful metric for identifying the transition to quantum-dominated magnetism in nanoscale systems.

To describe magnetism in such confined systems, new mathematical tools and physical principles are required. These include:

- Quantization of magnetic flux (e.g., $\Phi_0 = h/2e$),[5–19]
- Spin-based magnetism and Pauli matrices,[20,21]
- Magnetic dipole operators,[22] and
- Topological field theories that model phenomena such as quantized vortices and skyrmions.[23,24]

Maxwell's equations represent a continuum approximation that begins to lose predictive and descriptive power at quantum length scales. This necessitates a paradigm shift—from classical fields to quantum states—when modeling nanoscale or quantum-confined magnetic systems. The classical field notion, including $\nabla \cdot \mathbf{B} = 0$, must be reinterpreted or replaced altogether in the design and understanding of quantum magnetic devices and spintronic nanostructures.

If the ideal magnetic particle model holds, it becomes feasible to engineer arrays of nanoscale magnetic particles—such as 10 nm Fe_3O_4 nanoparticles—with sufficient inter-particle spacing to ensure magnetic isolation. In such configurations, each particle behaves independently, avoiding the collective dipole interactions that typically give rise to superparamagnetism. Instead, these particles may exhibit a distinct magnetic regime governed by internal dipole discreteness and quantum statistical effects. This opens the possibility of designing non-interacting magnetic arrays that do not conform to classical or superparamagnetic behavior, but instead manifest novel magnetic properties—potentially useful for quantum information storage, magnetic logic, or low-noise spintronic devices. Such systems could serve as a platform for exploring the transition between classical and quantum magnetism in a controlled and scalable manner.

Conclusion

In this work, we introduced the concept of the ideal magnetic particle—a theoretical model representing an isolated nanoscale magnetic system composed of a finite number of discrete dipoles. By systematically reducing the particle size below the critical threshold required to sustain a single quantum of magnetic flux, we demonstrated that the classical continuum approximation—and with it, the condition $\nabla \cdot \mathbf{B} = 0$—breaks down in a statistical sense. This breakdown is not driven by thermal fluctuations or inter-particle interactions, as in superparamagnetism, but arises fundamentally from the discrete and finite nature of the magnetic sources within the particle.

In this regime, the classical notion of continuous magnetic field lines becomes physically meaningless. Instead, magnetism must be described using quantum operators, flux quantization, and spin-based models. This conceptual shift mirrors the reinterpretation of electric field divergence at the quantum scale, where charge is no longer treated as a continuous distribution but as a quantized entity governed by wavefunctions and probability amplitudes.

It is important to clarify that the vector identity $\nabla \cdot B = 0$ remains mathematically exact at all length scales as a fundamental statement of Maxwell's equations. The discussion in this work does not imply that the law is violated in the nanoscale regime; rather, it highlights that the physical interpretation of B as a continuous, divergence-free flux density field becomes statistically unreliable when the number of contributing magnetic moments is very small. At the nanoscale, especially for isolated particles, the magnetic flux density ceases to be meaningfully defined in the continuum sense, and discrete quantum spin contributions dominate the magnetic response. Thus, the apparent "breakdown" refers to the loss of applicability of the continuous-field model, not to any mathematical failure of Maxwell's relation.

Our findings advocate for a paradigm shift in how magnetism is modeled at the nanoscale—from classical field-based descriptions to quantum state-based frameworks. This has significant implications for the design and understanding of quantum magnetic devices, spintronic nanostructures, and topological magnetic systems. The ideal magnetic particle model offers a foundational step toward this new understanding, providing a rigorous and conceptually consistent approach to magnetism in the quantum-confined limit.

Author contributions

D.S. conceived the theoretical concept, performed the mathematical calculations, analyzed the data, and wrote the manuscript.

Funding

This work was supported by the National Science Foundation under Grant No. CMMI-1953009.

Data availability

The data supporting the findings of this study are available in a publicly accessible repository.

Declarations

Conflict of interest

The author declares no conflict of interest.

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