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ON THE INVARIANCE PRINCIPLE UNDER MARTINGALE APPROXIMATION

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In this paper, we establish continuous Gaussian limits for stochastic processes associated to linear combinations of partial sums. The underlying sequence of random variables is supposed to admit a martingale approximation in the square mean. The results are useful in studying averages of additive functionals of a Markov chain with normal operator.

Keywords: Martingale approximation; normal Markov chains; functional CLT; Gaussian processes.

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1. Introduction

In recent years the approximation of a partial sum of a stationary process by a martingale with stationary differences was extensively studied. Papers by Dedecker, Merlevède and Volný [4], Zhao and Woodroofe [12], Gordin and Peligrad [7] and Peligrad [11], deal with necessary and sufficient conditions for the validity of a martingale approximation.

Many of these characterizations are strong enough for transporting from the martingale to the partial sums of a stationary sequence the central limit theorem, but fail to transport the central limit theorem in its functional form. In this paper, we show that an additional averaging of partial sums of a stationary sequence is useful in this respect. This modification allows us to obtain the weak convergence to a continuous Gaussian processes that we shall characterize via its covariance structure.

The result is easily applicable to stationary Markov chains with normal operators under a mild spectral condition that originates in the paper by Gordin and Lifshitz [6]. A simple example of normal Markov chain is a random walk on a compact group.

The paper is organized as follows. In Sec. 2, we give the definitions and formulate our results. In Sec. 3, we present an application to random walks on a compact group. Section 4 contains the proofs of theorems. The Appendix contains a generalized Toeplitz lemma.

2. Definitions and Results

2.1. *Notations*

We assume that $(\xi_n)_{n\in\mathbb{Z}}$ denotes a stationary and ergodic Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable space $(S, \mathcal{A})$. The marginal distribution is denoted by $\pi(A) = P(\xi_0 \in A)$ and we assume that there is a regular conditional distribution for ξ_1 given ξ_0 denoted by $Q(x, A) = \mathbb{P}(\xi_1 \in A|\, \xi_0 = x)$. Next let $\mathbb{L}_0^2(\pi)$ be the set of functions on S such that $\int f^2 d\pi < \infty$ and $\int f d\pi = 0$, and for a $f \in \mathbb{L}_0^2(\pi)$ let $X_i = f(\xi_i), S_n = \sum_{i=1}^n X_i$. Denote by $\mathcal{F}_k$ the σ-field generated by ξ_i with $i \leq k$. For any integrable variable X we denote $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$. In our notation $\mathbb{E}_0(X_1) = (Qf)(\xi_0) = \mathbb{E}(X_1|\xi_0)$. We denote by $\|X\|_2$ the norm in $\mathbb{L}_0^2(\Omega, \mathcal{F}, \mathbb{P})$ and also $\|f(\xi_0)\|_2 = \|f\|_2$, where the second notation stands for the norm in $\mathbb{L}_0^2(\pi)$.

In addition, Q denotes the operator on $\mathbb{L}_2(\pi)$ acting via $(Qf)(x) = \int_S f(s)Q(x, ds)$.

Notice that any stationary sequence $(X_k)_{k\in\mathbb{Z}}$ can be viewed as a function of a Markov process $\xi_k = (X_i; i \leq k)$, for the function $g(\xi_k) = X_k$.

The Markov chain is called normal if $QQ^* = Q^*Q$ on $\mathbb{L}^2(\pi)$. Then for every f in $\mathbb{L}^2(\pi)$ we denote by ρ_f the spectral measure of f with respect to Q on the closed unit disk that is uniquely determined by the relation

$$\int_S (Q^n f)(Q^m f) d\pi = \int_D z^m \bar{z}^n \rho_f(dz) \quad \text{with } n, m \geq 0.$$

2.2. *Results*

We say that $(X_n)_{n\in\mathbb{Z}}$ admits a martingale approximation in $\mathbb{L}_2$ if there is a martingale $(M_n)_{n\geq 1}$ with stationary differences $(D_n)_{n\in\mathbb{Z}}$ adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}}$ such that

$$\frac{1}{n}\mathbb{E}(S_n - M_n)^2 \longrightarrow 0, \tag{1}$$

where $S_n = \sum_{i=1}^n X_n$. If such an approximation exists it follows that the martingale is unique.

Relevant to our paper is the following characterization. According to Peligrad [11] representation (1) holds if and only if

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} [\mathbb{E}_0(S_l) - \mathbb{E}_{-1}(S_l)] = D_0 \text{ in } \mathbb{L}_2 \quad \text{and} \\ &\lim_{n\to\infty} \mathbb{E}(S_n^2)/n = \mathbb{E}(D_0^2) = \sigma^2. \end{aligned} \tag{2}$$

The approximation of type (1) is especially important for proving the central limit theorem. If the stationary sequence is ergodic, the martingale construction implies that the martingale differences are also stationary and ergodic and then, the limiting distribution of $S_n/\sqrt{n}$ is centered normal with the variance $\mathbb{E}(D_0^2)$.

To go further, let us introduce the notation $[x]$ as being the integer part of x and for $0 \le t \le 1$ define

$$W_n(t) = \frac{S_{[nt]}}{\sqrt{n}}.$$

$W_n(t)$ belongs to the space $D[0,1]$ of functions continuous from the right and having limits from the left that we endow with the uniform topology. An important problem with rich statistical applications is to study the convergence of $W_n(t)$ to the standard Brownian motion. A natural question is to investigate the limiting behavior of $W_n(t)$ for processes satisfying (1). In Proposition 4 of [4], an example was constructed showing that there are stationary sequences satisfying (2) and therefore (1) such that $W_n(t)$ is not tight in $D[0,1]$ endowed with the uniform topology. As a consequence, the convergence to a continuous process does not hold.

We shall see in this paper that continuous limits can be obtained for certain linear stochastic processes associated to partial sums.

For $0 \le \alpha < 1$ and a stochastic process $\mathbf{X} = (X_i)$ denote by

$$W_n^\alpha(t) = W_n^\alpha(t, \mathbf{X}) = \frac{1}{n^{3/2-\alpha}} \sum_{i=1}^{[nt]} \frac{S_i}{i^\alpha}. \tag{3}$$

Further, we denote by $Z_\alpha(t)$ a continuous Gaussian process with the covariance structure $(s \le t)$

$$\frac{s^{2-\alpha}}{(1-\alpha)(2-\alpha)} \left[t^{1-\alpha} - \frac{1}{3-2\alpha} s^{1-\alpha} \right]. \tag{4}$$

Our first result is a general statement.

Theorem 1. *Assume* (1) *holds. Then, for any* $0 \le \alpha < 1$

$$W_n^\alpha(t) \Longrightarrow |\sigma| Z_\alpha(t),$$

where σ^2 *is defined by* (2) *and* $\Rightarrow$ *denotes the weak convergence on* $D[0,1]$ *endowed with the uniform topology.*

By using this theorem, we study the additive functionals of a Markov chain with normal operator. As a corollary we obtain the following functional central limit theorem.

Theorem 2. *Assume* $(\xi_n)_{n\in\mathbb{Z}}$ *is a stationary ergodic Markov chain with* Q *normal and* $f \in \mathbb{L}_0^2(\pi)$ *satisfies*

$$\int_D \frac{1}{|1-z|} \rho_f(dz) < \infty. \tag{5}$$

Then the representation (1) *holds and for any* $0 \leq \alpha < 1$

$$W_n^\alpha(t) \Longrightarrow |\sigma| Z_\alpha(t).$$

We mention that the proof of the fact that (5) implies representation (1) can be found in Chap. 4, Sec. 7 of [2]. This chapter was written by Gordin and Lifshitz.

Denote by

$$V_n(f) = (I + Q + \cdots + Q^n)(f) \tag{6}$$

and then, with this notation, we have

$$\|\mathbb{E}_0(S_n)\|_2^2 = \|V_n(f)\|_2^2.$$

By Cuny [3, Lemma 2.1] an equivalent form of the condition (5) is

$$\sum_n \frac{\|V_n(f)\|_2^2}{n^2} < \infty. \tag{7}$$

3. Application to Random Walks on Compact Groups

In this section, we shall apply our result to random walks on compact groups.

Let $\mathcal{X}$ be a compact abelian group, $\mathcal{A}$ a σ-algebra of Borel subsets of $\mathcal{X}$ and π the normalized Haar measure on $\mathcal{X}$. The group operation is denoted by $+$. Let ν be a probability measure on $(\mathcal{X}, \mathcal{A})$. The random walk on $\mathcal{X}$ defined by ν is the Markov chain having the transition function

$$(x, A) \longrightarrow Q(x, A) = \nu(A - x).$$

The corresponding Markov operator, denoted also by Q, is defined by

$$(Qf)(x) = \int_{\mathcal{X}} f(x + y) Q(dy).$$

In this context,

$$(Q^* f)(x) = f * \nu^*(x) = \int_{\mathcal{X}} f(x - y) \nu(dy),$$

where ν^* is the image of ν by the map $x \to -x$. Then Q is a normal operator on $\mathbb{L}_\pi^2$. The dual group of $\mathcal{X}$, denoted by $\hat{\mathcal{X}}$, is discrete. Denote by $\hat{\nu}$ the Fourier transform of the measure ν, that is the function

$$g \longrightarrow \hat{\nu}(g) = \int_{\mathcal{X}} g(x) \nu(dx) \quad \text{with } g \in \hat{\mathcal{X}}.$$

A function $f \in \mathbb{L}^2(\pi)$ has the Fourier expansion

$$f = \sum_{g \in \hat{\mathcal{X}}} \hat{f}(g) g.$$

Ergodicity of Q is equivalent to $\hat{\nu}(g) \neq 1$ for any non-identity $g \in \hat{\mathcal{X}}$. By arguments in Borodin and Ibragimov [2, Chap. 4, Sec. 9] and also Derriennic and Lin [5, Sec. 8] condition (5) takes the form

$$\sum_{g\in\hat{\mathcal{X}}} \frac{|\hat{f}(g)|^2}{|1-\hat{Q}(g)|} < \infty. \tag{8}$$

Combining these considerations with our theorems, we obtain the following result:

Corollary 3. *Let Q be ergodic on $\mathcal{X}$. If for f in $\mathbb{L}_0^2(\pi)$ condition (8) is satisfied then the conclusion of Theorem 2 holds.*

4. Proofs

We establish first the validity of Theorem 1 for martingales.

Proposition 4. *Let $(M_i)_{i\in\mathbb{Z}}$ be a martingale with stationary and ergodic differences $(D_i)_{i\in Z}$ on $(\Omega,\mathcal{F},\mathbb{P})$ adapted to an increasing filtration of sub-sigma algebras of $\mathcal{F}$, $(\mathcal{F}_i)_{i\in\mathbb{Z}}$. Assume $\mathbb{E}(D_0^2)=1$ and let $0\le\alpha<1$. Then $W_n^\alpha(t,\mathbf{D})$ defined by (3) converges weakly to $Z_\alpha(t)$, where $Z_\alpha(t)$ is a continuous Gaussian process with the covariance structure (4).*

Proof. The proof involves several steps and is based on Theorem 13.5 of [1]. We shall rewrite the process as a linear process with simplified coefficients. The coefficients will be continuous functions of t. Next step is to establish tightness in $D[0,1]$ for $W_n^\alpha(t,\mathbf{D})$ followed by the convergence of finite dimensional distributions to the corresponding ones of $Z_\alpha(t)$.

(1) *Asymptotic equivalence*

For $0\le s\le 1$, we start by rewriting $W_n^\alpha(s)=W_n^\alpha(s,\mathbf{D})$ as a linear process with stationary and ergodic martingale innovations.

$$W_n^\alpha(s)=\frac{1}{n^{3/2-\alpha}}\sum_{j=1}^{[ns]}\frac{M_j}{j^\alpha}=\frac{1}{n^{3/2-\alpha}}\sum_{i=1}^{[ns]}\left(\sum_{j=i}^{[ns]}\frac{1}{j^\alpha}\right)D_i=\sum_{i=1}^{[ns]}c_{n,i}(s)D_i,$$

where

$$c_{n,i}(s)=\frac{1}{n^{3/2-\alpha}}\sum_{j=i}^{[ns]}\frac{1}{j^\alpha},\quad 1\le i\le[ns].$$

It is convenient to smooth the sequence of constants. We consider now the random element

$$W_n'(s)=\sum_{i=1}^{[ns]}c_{n,i}'(s)D_i,$$

where for $1\le i\le[ns]$,

$$c_{n,i}'(s)=\frac{1}{(1-\alpha)n^{3/2-\alpha}}((ns)^{1-\alpha}-i^{1-\alpha}) \tag{9}$$

and observe that by the stationarity of the martingale differences we have

$$\sup_{0\le s\le 1}|W_n^{\alpha}(s)-W_n'(s)|\le \frac{1}{n^{3/2-\alpha}}\sum_{i=1}^{n}\left(\sum_{j=i}^{n} j^{-\alpha}-(j+1)^{-\alpha}\right)|D_i|$$

$$\le \frac{\max_{1\le i\le n}|D_i|}{n^{3/2-\alpha}}\sum_{j=1}^{n} j^{-\alpha}$$

$$\le \frac{\max_{1\le i\le n}|D_i|}{(1-\alpha)n^{1/2}}\longrightarrow 0 \text{ a.s. and in } \mathbb{L}_2.$$

Therefore by Theorem 3.1 in [1] we can analyze instead of the process $W_n^{\alpha}(s)$, the process $W_n'(s)$.

(2) *Tightness in* $D(0,1)$

Let $0\le s<t\le 1$. We estimate the second moment of an increment

$$\mathbb{E}(W_n'(t)-W_n'(s))^2=\mathbb{E}\left(\sum_{i=1}^{[nt]} c_{n,i}'(t)D_i-\sum_{i=\mathbf{1}}^{[ns]} c_{n,i}'(s)D_i\right)^2$$

$$\le 2\sum_{i=1}^{[ns]}(c_{n,i}'(t)-c_{n,i}'(s))^2+2\sum_{i=[ns]+1}^{[nt]}(c_{n,i}'(t))^2=I+II.$$

Since for a constant $c=c(\alpha)$

$$I\le c(t^{1-\alpha}-s^{1-\alpha})^2$$

and

$$II\le \frac{c}{n^{3/2-\alpha}}\sum_{i=[ns]+1}^{[nt]}((nt)^{1-\alpha}-i^{1-\alpha})^2\le c(t^{1-\alpha}-s^{1-\alpha})^2,$$

by combining the estimates, we can find a constant c' such that for all $n\ge 1$

$$\mathbb{E}(W_n'(t)-W_n'(s))^2\le c'(t^{1-\alpha}-s^{1-\alpha})^2.$$

Now by applying Theorem 11.6 in [9] with $d(s,t)=|s^{1-\alpha}-t^{1-\alpha}|$ we conclude that for each $\varepsilon>0$ there is $\eta>0$ such that

$$\mathbb{E}\sup_{d(s,t)<\eta}|W_n'(t)-W_n'(s)|<\varepsilon,$$

and the tightness in $D[0,1]$ endowed with the uniform topology follows, since $d(s,t)\le |t-s|^{\alpha}$.

(3) *Estimation of the limiting covariances*

For $s\le t$,

$$\operatorname{cov}(W_n'(s),W_n'(t))=\sum_{i=1}^{[ns]} c_{n,i}'(s)c_{n,i}'(t).$$

By $x_n \sim y_n$ we understand that $\lim_{n\to\infty} x_n/y_n = 1$. By elementary calculus

$$\mathrm{cov}(W_n'(s), W_n'(t)) \sim \frac{s^{2-\alpha}}{(1-\alpha)(2-\alpha)}\left(t^{1-\alpha} - \frac{1}{(3-2\alpha)}s^{1-\alpha}\right) \quad \text{as } n \longrightarrow \infty$$

and then

$$\mathrm{var}(W_n'(t)) \sim \frac{2t^{3-2\alpha}}{(2-\alpha)(3-2\alpha)} \quad \text{as } n \longrightarrow \infty.$$

(4) *Convergence of the finite dimensional distributions*

We establish now the convergence of finite dimensional distributions of $W_n'(t)$ to the corresponding ones of $Z_\alpha(t)$. It is convenient to extend the definition of the coefficients $c_{n,i}(s)$ defined in (9) beyond the range $1 \le i \le [ns]$. We define $c_{n,i}(s) = 0$ for $[ns] + 1 \le i \le n$.

Let l be a fixed integer and let $(\lambda_k)_{1\le k\le l}$ be a vector of real numbers. We have to study the limiting distribution of

$$\sum_{k=1}^{l} \lambda_k W_n'(t_k) = \sum_{i=1}^{n}\left[\sum_{k=1}^{l} \lambda_k c_{n,i}'(t_k)\right] D_i.$$

It is enough to show that the limit is Gaussian and the limiting covariance structure, already determined at point 3 of the proof, is actually the covariance structure of the limiting distribution.

We shall verify the conditions for the central limit theorem from Hall and Heyde [8, Theorem 3.1] for the triangular array of martingale differences: $\Delta_{n,i} = [\sum_{k=1}^{l} \lambda_k c_{n,i}'(t_k)]D_i$.

First, we notice that by (9) we have for a certain positive constant K depending only of α, k, and $(\lambda_k)_{1\le k\le l}$

$$\max_{1\le i\le n} |\Delta_{n,i}| \le K \max_{1\le i\le n} \frac{|D_i|}{n^{1/2}}.$$

By taking into account the stationarity of $(D_i)_{i\ge 0}$ we obtain

$$\max_{1\le i\le n} |\Delta_{n,i}| \longrightarrow 0 \quad \text{in } \mathbb{L}_2. \tag{10}$$

Then, we show that the sequence $\sum_{i=1}^{n} \Delta_{n,i}^2$ converges almost surely and in $\mathbb{L}_1$. Notice that

$$\begin{aligned}\sum_{i=1}^{n} \Delta_{n,i}^2 &= \sum_{i=1}^{n}\sum_{k=1}^{l} \lambda_k^2 (c_{n,i}'(t_k))^2 D_i^2 \\ &\quad + 2\sum_{i=1}^{n}\left[\sum_{k=1}^{l-1}\sum_{j=k+1}^{l} \lambda_k\lambda_j c_{n,i}'(t_k)c_{n,i}'(t_j)\right] D_i^2.\end{aligned}$$

It is enough to show the existence and to compute the following limit, for all $0 \leq s \leq t \leq 1$:

$$\lim_{n\to\infty} \sum_{i=1}^{[ns]} c'_{n,i}(s)c'_{n,i}(t)D_i^2.$$

We have

$$(1-\alpha)^2 c'_{n,i}(s)c'_{n,i}(t) = \frac{(st)^{1-\alpha}}{n} + \frac{i^{2-2\alpha}}{n^{3-2\alpha}} - \frac{s^{1-\alpha}i^{1-\alpha}}{n^{2-\alpha}} - \frac{t^{1-\alpha}i^{1-\alpha}}{n^{2-\alpha}}.$$

By the ergodic theorem we know that

$$\frac{1}{n}\sum_{i=1}^{[ns]} D_i^2 \longrightarrow s \quad \text{a.s. and in } \mathbb{L}_1.$$

By Lemma A and via Remark A in the Appendix, applied first with $c_i = i^{2-2\alpha}$ and then with $c_i = i^{1-\alpha}$ we obtain

$$\frac{1}{n^{3-2\alpha}}\sum_{i=1}^{[ns]} i^{2-2\alpha}D_i^2 \longrightarrow \frac{s^{3-2\alpha}}{3-2\alpha} \quad \text{a.s. and in } \mathbb{L}_1$$

and

$$\frac{1}{n^{2-\alpha}}\sum_{i=1}^{[ns]} i^{1-\alpha}D_i^2 \longrightarrow \frac{s^{2-\alpha}}{2-\alpha} \quad \text{a.s. and in } \mathbb{L}_1.$$

It follows that $\sum_{i=1}^{n} \Delta_{n,i}^2$ converges almost surely and in $\mathbb{L}_1$ and the limiting distribution of $\sum_{k=1}^{l} \lambda_k W'_n(t_k)$ is therefore normal. In particular, for all $0 \leq t \leq 1$ the limiting distribution of $W'_n(t)$ is centered normal with variance $2t^{3-2\alpha}/(2-\alpha)(3-2\alpha)$ and in addition $[W'_n(t)]^2$ is uniformly integrable. It follows that the covariance structure determined at point 3 of the proof is indeed the covariance structure of the limiting distribution.

Proof of Theorem 1. In order to prove Theorem 1 we use the estimate

$$\left\| \sup_{0\leq t\leq 1} |W_n^\alpha(t,\mathbf{X}) - W_n^\alpha(t,\mathbf{D})| \right\|_2 \leq \frac{1}{n^{3/2-\alpha}} \sum_{k=1}^{n} \frac{\|S_k - M_k\|_2}{k^\alpha}.$$

We notice that (1) implies that

$$\frac{1}{n^{2-2\alpha}} \sum_{k=1}^{n} \frac{\|S_k - M_k\|_2^2}{k^{2\alpha}} \longrightarrow 0.$$

Then, by Cauchy–Schwarz inequality

$$\sum_{k=1}^{n} \frac{\|S_k - M_k\|_2}{k^\alpha} \leq \left[n \sum_{k=1}^{n} \frac{\|S_k - M_k\|_2^2}{k^{2\alpha}} \right]^{1/2}.$$

Therefore

$$\left\| \sup_{0\le t\le 1} |W_n^\alpha(t,\mathbf{X}) - W_n^\alpha(t,\mathbf{D})| \right\|_2 \longrightarrow 0.$$

By Theorem 3.1 in [1], the process $W_n^\alpha(t,\mathbf{X})$ has the same limiting distribution as $W_n^\alpha(t,\mathbf{D})$. Then, we just have to apply Proposition 4 which is valid for stationary and ergodic martingale differences to obtain the result. □

Appendix

We give here a generalized Toeplitz lemma. We did not find it in the literature so we include it with a proof.

Lemma A. *Assume $(x_i)_{i\ge 1}$ and $(c_i)_{i\ge 1}$ are sequences of real numbers such that*

$$\frac{1}{n}\sum_{i=1}^n x_i \longrightarrow L, \quad nc_n \longrightarrow \infty \quad \textit{and} \quad \frac{c_1+\cdots+c_n}{nc_n} \to C < 1.$$

Then,

$$\frac{\sum_{i=1}^n c_i x_i}{\sum_{i=1}^n c_i} \longrightarrow L.$$

Proof. Denote

$$A_n = \frac{1}{n}\sum_{i=1}^n x_i.$$

It follows that $x_i = iA_i - (i-1)A_{i-1}$ and by Abel summation,

$$\sum_{i=1}^n c_i x_i = \sum_{i=1}^n c_i(iA_i - (i-1)A_{i-1}) = c_n n A_n + \sum_{i=1}^{n-1}(c_i - c_{i+1}) iA_i.$$

Notice that

$$\begin{aligned}\sum_{i=1}^{n-1}(c_{i+1}-c_i)i &= c_n(n-1) - (c_1+c_2+\cdots+c_{n-1})\\ &\sim (1-C)c_n n \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.\end{aligned}$$

By Toeplitz lemma

$$\frac{\sum_{i=1}^{n-1}(c_i-c_{i+1})iA_i}{\sum_{i=1}^{n-1}(c_{i+1}-c_i)i} \longrightarrow -L \quad \text{as } n \longrightarrow \infty.$$

So

$$\frac{\sum_{i=1}^{n-1}(c_i-c_{i+1})iA_i}{(1-C)c_n n} \longrightarrow -L \quad \text{as } n \longrightarrow \infty.$$

Therefore

$$\frac{\sum_{i=1}^{n} c_i x_i}{c_n n} = A_n + \frac{1}{c_n n} \sum_{i=1}^{n-1} (c_i - c_{i+1}) i A_i \longrightarrow L - (1 - C)L = CL$$

and the result follows. □

Remark A. We notice that Lemma A is applicable for the sequences of constants $c_i = i^{2-2\alpha}$ and $c_i = i^{1-\alpha}$ for some $0 \le \alpha < 1$.

Proof. By Riemann sum arguments

$$\frac{1 + i^{2-2\alpha} + \cdots + n^{2-2\alpha}}{n^{3-2\alpha}} \longrightarrow \frac{1}{3 - 2\alpha} < 1$$

and

$$\frac{1 + i^{1-\alpha} + \cdots + n^{1-\alpha}}{n^{2-2\alpha}} \longrightarrow \frac{1}{2 - \alpha} < 1.$$ □

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