A BIHOLOMORPHISM FROM THE BELL REPRESENTATIVE DOMAIN ONTO AN ANNULUS AND KERNEL FUNCTIONS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Mustafa Ersin Değer

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

August 2007

Purdue University

West Lafayette, Indiana

Anne ve Babama...

doğacak çocuklarıma...

Eşime ve kardeşime...

ACKNOWLEDGMENTS

This thesis could not have been successful without the help of so many around me. First and foremost, I would like to thank my advisor Prof. Steve Bell for introducing me to this thesis topic and helping me throughout my graduate study at Purdue. Also, thanks to my committee members Prof. David Catlin, Prof. László Lempert and Prof. Allen Weitsman.

I have received the help of a number of friends of mine, only some of whom are Raif Rustamov, George Hassapis, Cemil Selçuk and Kuan Tan. Last but not least, I would like to thank my wife, mother and father who endured this long and tiresome process with me, always offering support and love. I hope and believe that I have been able to accomplish a bit of what I was planning.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

Değer, Mustafa Ersin Ph.D., Purdue University, August, 2007. A biholomorphism from the Bell representative domain onto an annulus and Kernel functions . Major Professor: Steven R. Bell.

Let $A_{\rho^2,1} = \{z \in \mathbb{C} : \rho^2 < |z| < 1\}$ and let $\Omega_r = \{z \in \mathbb{C} : |z + \frac{1}{z}| < r\}$. It is known that for $r > 2$, Ω_r is a doubly-connected domain with an algebraic Bergman kernel and satisfies a certain quadrature identity. We find an explicit biholomorphism between these two domains, and get the value of ρ as a function of r, which follows from the computation of the biholomorphism. Using the transformation formula for the Bergman kernel, we write out the Bergman kernel of the Bell representative domain Ω_r, which was earlier known to be algebraic. We also determine the Ahlfors maps of this generalized quadrature domain using the Ahlfors map of the annulus.

1. Introduction

1.1 Kernel functions

The Bergman and Szegö kernels are important objects of geometric function theory and complex analysis. The Bergman kernel $K(z, w)$ of a domain Ω has the property $\int_\Omega f(w)K(z, w)\, dV_w = f(z)$ when f is a square-integrable holomorphic function in Ω. Thus, the Bergman kernel is sometimes called the reproducing kernel for the set of square-integrable holomorphic functions on Ω. The Szegö kernel $S(z, w)$ of Ω has a similar reproducing property $\int_{\partial\Omega} f(w)S(z, w)\, ds_w = f(z)$, where f is any holomorphic function on Ω that is square-integrable on $\partial\Omega$, the boundary of Ω. The Bergman and Szegö kernels are holomorphic in the first variable and antiholomorphic in the second on $\Omega \times \Omega$, and they are hermitian, e.g. $K(w, z) = \overline{K(z, w)}$. Here, dV and ds denote the volume element and the arc-length measure, respectively.

A quantity of remarkable results were achieved by the use of these kernels in the theory of functions of one and several complex variables, in conformal mappings, in invariant Riemannian metrics and in other subjects [1]. The applications of the kernels for solving boundary value problems of PDEs of elliptic type have been quite fruitful as well; and there is a number of recent discoveries obtained using the kernel functions in the field of Fluid Dynamics [7].

Another advantage of working with the Bergman kernel functions is the possibility of generalizing some strong and important results on holomorphic functions of one variable to those of several variables (such as the Schwarz's lemma). In one complex variable, we have the option of representing a number of important conformal mappings in terms of the kernel functions. For instance, for a simply connected domain Ω

and z_0 in Ω, the function $\xi = f(z)$ which maps Ω onto a disk $|\xi| < R$ with $f(z_0) = 0$ and $f'(z_0) = 1$ is given by [1]

$$f(z) = \frac{1}{K(z_0, z_0)} \int_{z_0}^{z} K(s, z_0)\, ds\,.$$

1.2 The Ahlfors map

This brings us to the concept of the Riemann mapping function and the Ahlfors map. Let us denote the unit disk by $\mathbb{D}$. We all know that the Riemann mapping function can be regarded as the solution to the following extremal problem: For a simply connected domain $\Omega \neq \mathbb{C}$ and fixed $b \in \Omega$, construct a holomorphic map $f_b : \Omega \to \mathbb{D}$ with $f_b'(b) > 0$ and as large as possible. The Riemann map is the solution to this problem. It is the unique conformal, 1-1 and onto map $f_b : \Omega \to \mathbb{D}$ with $f_b'(b) > 0$ and $f_b(b) = 0$.

For *multiply connected* domains Ω of connectivity $n > 1$, the answer to the same extremal problem above becomes the Ahlfors map. It is the unique holomorphic map $f_b : \Omega \to \mathbb{D}$ that is onto, $f_b'(b) > 0$ and $f_b(b) = 0$ (see [2], pp 48-49). However, it has $2n - 2$ branch points in the interior and is no longer 1-1 there. In fact, it maps Ω onto $\mathbb{D}$ in an $n : 1$ fashion, and maps each boundary curve 1-1 onto the unit circle $\mathbb{T}$. Therefore, the Ahlfors map can be regarded as the "Riemann mapping function for multiply connected domains". It also takes on this same role from the viewpoint of the kernel functions.

1.3 Bell representative domains

Let $\Omega_r = \{z \in \mathbb{C} : |z + \frac{1}{z}| < r\}$, and consider this domain only for $r > 2$. Jeong and Taniguchi (see [9]) called this domain to be a Bell representation in the plane and that is where we get our motivation for referring to it as the *Bell representative domain* from now on. It is a classical fact that any doubly connected domain, none of whose boundary components reduces to a point, is biholomorphic to an annulus. Therefore, we know *a priori* that the 2-connected Bell representative domain is biholomorphic to an annulus. However, finding the explicit formula for the biholomorphism is a major task.

Of much interest here, the Bell representative domain has the virtue that all the classical domain functions associated to it are algebraic: these are the Bergman kernel, Szegö kernel, the Ahlfors map, and the Poisson kernel, among others [4]. Furthermore, every doubly connected domain in the plane is biholomorphic to precisely one Bell representative domain [4]. So, this domain may serve as a representative domain for doubly connected planar domains, since its kernel functions are much easier to work with than other two-connected domains.

In regards to its shape, the Bell representative domain looks like a distorted annulus, symmetrical with respect to the real and imaginary axes as well as with respect to the origin (see Fig 2.1). The unit circle $\mathbb{T}$ always lies inside the Bell representative domain, since $e^{i\theta} + e^{-i\theta} = 2\,cos\,\theta \;\leq\; 2 \;<\; r$ for any real θ.

Here is an outline of our results and the way this thesis is organized:

In Chapter Two, we compute an explicit biholomorphic map from the Bell representative domain onto an annulus. We investigate some of the properties of this map which are of crucial importance for the next chapters. Our way of finding the biholomorphic equivalence between the annulus and the Bell representative domain differs from that of [9] in which we write down the biholomorphism *explicitly*, use elementary arguments and well-known conformal mappings and we find the modulus of the annulus in terms of the parameter of the Bell representative domain. (Compare also with [9].)

In Chapter Three, we determine the Ahlfors maps of the Bell representative domain. We locate its zeroes (and, hence locate that of S_{Ω_r}) as well as notice that the branch points lie on the unit circle for any given Ahlfors map. We are also able to find explicit formulas for it based at any point $z_0 \in \Omega_r$.

In Chapter Four, we reach the most important result of this thesis where we find the Bergman kernel of the Bell representative domain. We compute it in explicit form and show that it is algebraic. (It was shown in [4] that it has an algebraic Bergman kernel function, but no closed formula was given there.) Our discovery has a wider significance and general implications than the sheer computation of the kernel function, which itself is remarkable.

In Chapter Five, we point out some questions and mention some areas for future exploration.

2. Biholomorphism from Ω_r onto an annulus

2.1 Dividing Ω_r into pieces

Let us denote the unit disc by $\mathbb{D}$. Define:

$$\begin{aligned}
\Omega^1 &= \Omega_r \cap \text{LHP} \cap \mathbb{D} \\
\Omega^2 &= \Omega_r \cap \text{UHP} \cap \mathbb{D} \\
\Omega^3 &= (\Omega_r \cap \text{LHP}) - \mathbb{D} \\
\Omega^4 &= (\Omega_r \cap \text{UHP}) - \mathbb{D}
\end{aligned}$$

where UHP and LHP denote the upper-half plane $\{z : \text{Im}(z) > 0\}$ and the lower half-plane $\{z : \text{Im}(z) < 0\}$, respectively. Introduce a new parameter $0 < k < 1$ by

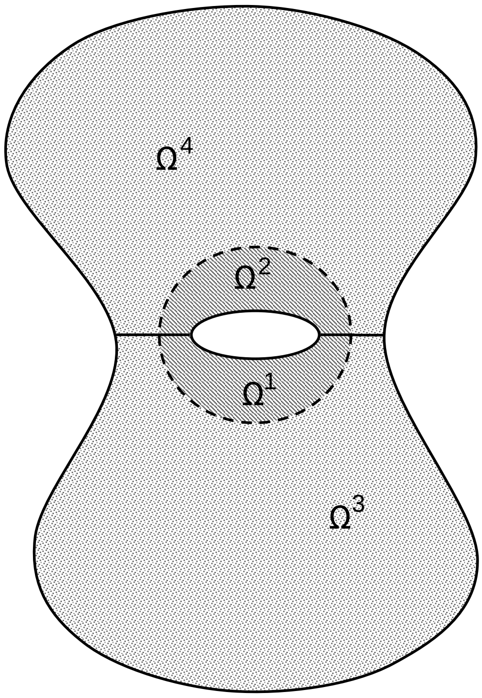

Figure 2.1. The Bell representative domain Ω_r.

$r = \frac{2}{\sqrt{k}}$. We will first find a biholomorphic map from Ω^1 above onto a part of the annulus, and then extend it analytically throughout Ω_r by reflection.

Let us consider the conformal map of the upper half plane $\{z : \mathrm{Im}(z) > 0\}$ onto a rectangle in the w-plane, asserting that the points $z = \pm 1$ and $z = \pm\frac{1}{k}$ are mapped to the corners of the rectangle in the w-plane.

In view of the Schwarz-Christoffel formula, this mapping [13] is given by the function

$$w = F(z) = \int_0^z \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}.$$

We take the branch of the square root function above so that $\sqrt{1} = 1$. We can find out the location of the rectangle: If you stay on the real axis around the origin, notice that $F(-z) = -F(z)$ as well as $F(z)$ is real. Consequently, we deduce that the rectangle is situated symmetrically with respect to the imaginary axis and that one of its sides lies on the real axis. It is customary to denote the height by K' and the width by $2K$ of this rectangle. In summary,

$$F(-\frac{1}{k}) = -K + iK', \quad F(-1) = -K, \quad F(1) = K, \quad F(\frac{1}{k}) = K + iK'$$

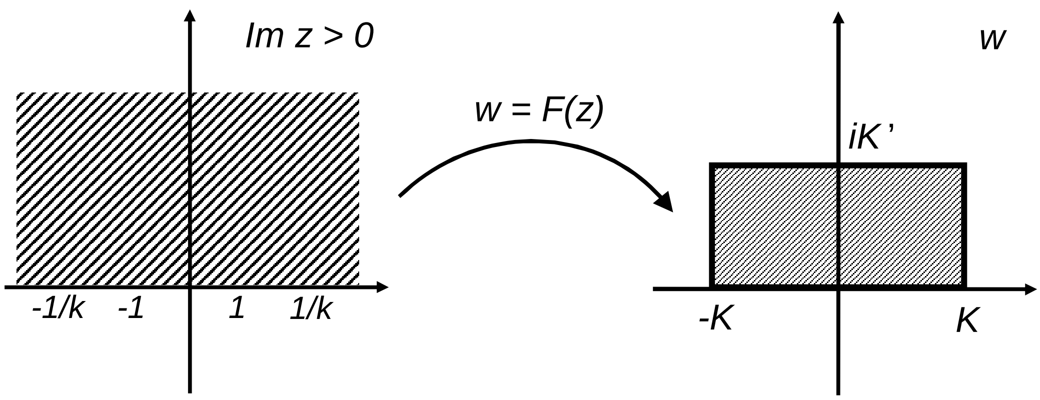

Figure 2.2. $F(z)$ maps the upper half-plane onto a rectangle as shown.

Using the definition for K and after some manipulation and a change of variable for K', we obtain

$$
\begin{aligned}
K &= \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \\
K' &= \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k'^2\xi^2)}}
\end{aligned}
$$

where $k' = \sqrt{1-k^2}$.

Many books when introducing the theory of elliptic functions adopt a similar approach [13] such as the following to define the *Jacobi elliptic functions*: Define $z = \operatorname{sn} w$ as the holomorphic function that is inverse to the function F defined above. Clearly, sn depends also on the parameter k. We may write $sn(w,k)$ to indicate this relationship. One can extend $sn(w,k)$ beyond the rectangle described above to the whole complex plane to obtain a doubly periodic function also known as the Jacobi sn-function. We will use the terms "Jacobi sn-function" and "elliptic sine function" interchangeably. In literature, F is also called an "elliptic integral". Due to the fact that F could be regarded as the inverse to the Jacobi sn-function, we will from now on use the notation $F(z) = \operatorname{sn}^{-1}(z,k)$.

2.2 Building the biholomorphism

Now we build a conformal map from Ω^1 onto the lower half annulus $A^-_{\rho,1}$, where $A^-_{\rho,1} = A_{\rho,1} \cap \text{LHP}$, where we make sure the part of the unit circle on the boundary of Ω^1 is fixed, since this will be very important for the rest of the mapping process.

First, consider $J : \Omega^1 \to D^+_{1/\sqrt{k}}$. Here $J(z) = \frac{1}{2}(z + \frac{1}{z})$ is the Joukowski map and $D^+_{1/\sqrt{k}}$ is part of the disk of radius $\frac{1}{\sqrt{k}}$ centered at the origin which lies on the upper half plane (see Fig 2.3). Clearly, J is conformal, 1-1 and onto.

Second, consider $F : D^+_{1/\sqrt{k}} \to R$ where R is the rectangle with corners at the points $-K, K, K + i\frac{K'}{2}, -K + i\frac{K'}{2}$. We prove that F is a conformal and onto map.

Proposition 2.2.1 *$z \mapsto F(z)$ is a conformal mapping from the half-disk $D^+_{1/\sqrt{k}}$ onto R, a rectangle with corners at the points $-K,\ K,\ K + i\frac{K'}{2}$ and $-K + i\frac{K'}{2}$.*

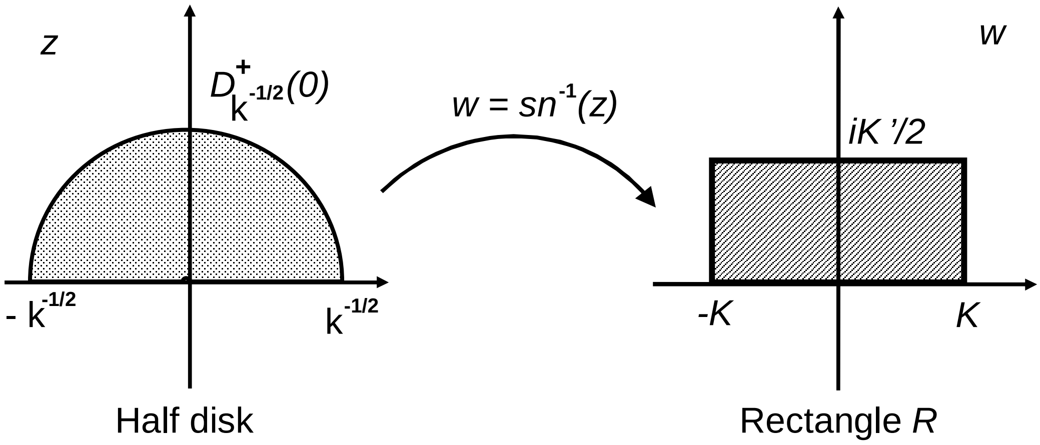

Figure 2.3. The elliptic integral $\mathrm{sn}^{-1}(.)$ maps $D^+_{1/\sqrt{k}}$ onto a rectangle R.

Proof It suffices to prove that the elliptic sine function sn maps this rectangle onto $D^+_{1/\sqrt{k}}$. Introduce the elliptic functions

$$\mathrm{cn}\, z = \sqrt{1 - \mathrm{sn}^2 z}, \quad \mathrm{cn}\, 0 = 1$$

as well as

$$\mathrm{dn}\, z = \sqrt{1 - k^2 \mathrm{sn}^2 z}, \quad \mathrm{dn}\, 0 = 1.$$

We have the following addition formula for the sn-function:

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn} u\, \mathrm{cn} v\, \mathrm{dn} v \ + \ \mathrm{sn} v\, \mathrm{cn} u\, \mathrm{dn} u}{1 - k^2 \mathrm{sn}^2 u\, \mathrm{sn}^2 v}$$

Considering the mapping properties of sn, we already know that the segment $[-K, K]$ is mapped to $[-1, 1]$. Using $\mathrm{sn}(\frac{iK'}{2}) = \frac{i}{\sqrt{k}}$, $\mathrm{cn}(\frac{iK'}{2}) = \frac{\sqrt{1+k}}{\sqrt{k}}$ and $\mathrm{dn}(\frac{iK'}{2}) = \sqrt{1+k}$ as well as the sum formula above for $u = \pm K$ and $v = \frac{iK'}{2}$, we conclude that $\mathrm{sn}(-K + \frac{iK'}{2}) = -\frac{1}{\sqrt{k}}$ and $\mathrm{sn}(K + \frac{iK'}{2}) = \frac{1}{\sqrt{k}}$ (see [8, 12]). By continuity, this means the vertical line segment from $-K$ to $-K + \frac{iK'}{2}$ is mapped onto the real line $[-\frac{1}{\sqrt{k}}, -1]$. Similarly, the vertical line segment from K to $K + \frac{iK'}{2}$ is mapped onto the real line $[1, \frac{1}{\sqrt{k}}]$.

Now consider the line segment $\alpha K + \frac{iK'}{2}$ where $-1 \le \alpha \le 1$. It suffices to prove $|\mathrm{sn}(\alpha K + \frac{iK'}{2})| = \frac{1}{\sqrt{k}}$, since that will mean that $\mathrm{sn}(\alpha K + \frac{iK'}{2})$ lies on $\{z : |z| = \frac{1}{\sqrt{k}}$ and $\mathrm{Im} z > 0\}$. (Also see [10, 17]). Consequently, sn will be a conformal map from R onto $D^+_{1/\sqrt{k}}$ and F will be its inverse.

We have:

$$\begin{aligned}
\mathrm{sn}(\alpha K + \frac{iK'}{2}) &= \frac{\mathrm{sn}(\alpha K)\,\mathrm{cn}(\frac{iK'}{2})\,\mathrm{dn}(\frac{iK'}{2}) \;+\; \mathrm{sn}(\frac{iK'}{2})\,\mathrm{cn}(\alpha K)\,\mathrm{dn}(\alpha K)}{1 - k^2\mathrm{sn}^2(\alpha K)\,\mathrm{sn}^2(\frac{iK'}{2})} \\
&= \frac{\beta\,\frac{1+k}{\sqrt{k}} \;+\; \frac{i}{\sqrt{k}}\,\sqrt{1-\beta^2}\,\sqrt{1-k^2\beta^2}}{1 \;-\; k^2\,\beta^2\,(-\frac{1}{k})}, \quad -1 \le \beta \le 1 \\
&= \frac{1}{\sqrt{k}}\{\frac{\beta(1+k) + i\sqrt{(1-\beta^2)(1-k^2\beta^2)}}{1+k\beta^2}\}
\end{aligned}$$

It is easy now to check that

$$\begin{aligned}
|\frac{\beta(1+k) + i\sqrt{(1-\beta^2)(1-k^2\beta^2)}}{1+k\beta^2}|^2 &= \frac{\beta^2(1+2k+k^2) + (1-\beta^2)(1-k^2\beta^2)}{1+2k\beta^2+k^2\beta^4} \\
&= \frac{\beta^2 + 2k\beta^2 + k^2\beta^2 + 1 - \beta^2 - k^2\beta^2 + k^2\beta^4}{1+2k\beta^2+k^2\beta^4} \\
&= 1
\end{aligned}$$

■

Next, we consider the action of the map $\omega : z \mapsto \frac{\pi i}{2K} z$ on the rectangle R above, and then the map $g : z \mapsto -ie^z$ from the rotated rectangle. Now, choose ρ such that $\log \rho = -\frac{\pi K'}{4K}$. This choice of ρ constitutes the relationship between k and ρ, and therefore gives ρ as a function of r.

The sequence of the maps above helps us reach $A^-_{\rho,1} = A_{\rho,1} \cap \mathrm{LHP}$. It is straightforward to check that the composition of the maps is given by

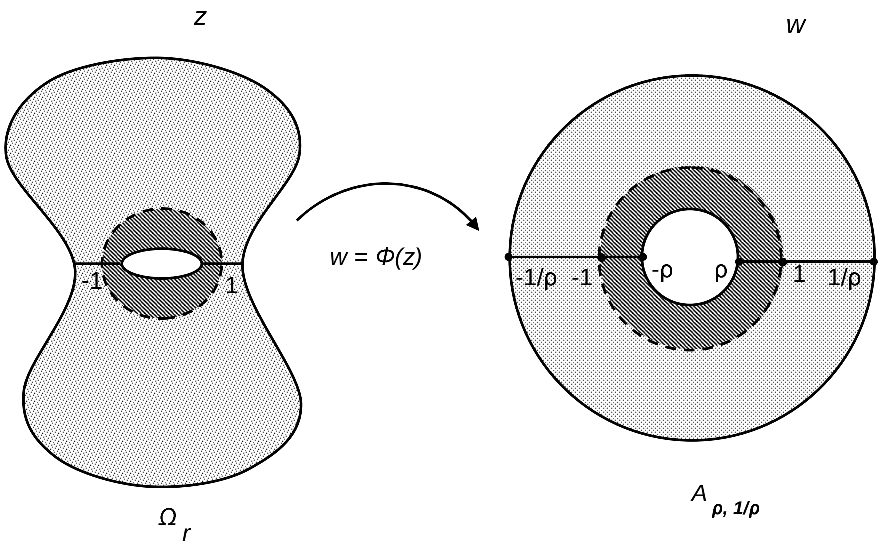

Figure 2.4. Φ maps the Bell representative domain Ω_r onto the annulus $A_{\rho,1/\rho}$.

$$(g \circ \omega \circ F \circ J)(z) = -i\, e^{\frac{i\pi}{2K}\operatorname{sn}^{-1}(\frac{1}{2}(z+1/z))}.$$

Let us restate this result:

Proposition 2.2.2 *The composition map $\Phi : \Omega^1 \to A^-_{\rho,1}$ is conformal, 1-1 and onto. This mapping is explicitly given by $\Phi(z) = -i\, e^{\frac{\pi i}{2K}\operatorname{sn}^{-1}(\frac{1}{2}(z+1/z))}$*

This map Φ has a number of important properties, most of which will be central in our future computations:

1. Since $\Phi : \Omega^1 \to A^-_{\rho,1}$ sends the real axis to the real axis, it extends analytically to $\Omega^2 = \Omega_r \cap \text{UHP} \cap \mathbb{D}$ by Schwarz reflection principle via the rule $\Phi(z) = \overline{\Phi(\bar{z})}$.

2. The mapping $\Phi : \Omega^1 \to A^-_{\rho,1}$ also satisfies that $-\Phi(z) = \Phi(-z)$ if z lies on the real axis. Consequently, this gives us yet another extension onto Ω^2.

3. The extended mapping Φ on $\overline{\Omega^1 \cup \Omega^2}$ fixes the unit circle.

Let us prove the second assertion, since the first and the third are trivial from the construction of Φ.

Proposition 2.2.3 *The conformal map $\Phi : \Omega^1 \to A^-_{\rho,1}$ defined as $\Phi(z) = -i\, e^{\frac{\pi i}{2K}\operatorname{sn}^{-1}(\frac{1}{2}(z+1/z))}$ is an odd function on the real line segment of the boundary of Ω^1: $\Phi(z) = -\Phi(-z)$.*

Proof Pick a point x on the real line and on the boundary of Ω^1. Without loss of generality, suppose $x > 0$. Clearly, $\mathrm{sn}^{-1}(J(x)) = K + i\alpha K'$ for some $\alpha = \alpha(x)$ with $0 \le \alpha \le \frac{1}{2}$.

We claim that $\mathrm{sn}^{-1}(J(-x)) = -K + i\alpha K'$.

Proof of claim. Let $u = \mathrm{sn}^{-1}(J(-x)) = \mathrm{sn}^{-1}(-J(x))$ and $v = \mathrm{sn}^{-1}(J(x))$. Then $\mathrm{sn}\, u = -J(x)$ and $\mathrm{sn}\, v = J(x)$, so that $\mathrm{sn}\, u + \mathrm{sn}\, v = 0$. We have $v = K + i\alpha K'$ and $u = -K + i\gamma K'$ and we will show $\gamma = \alpha$.

By [12] on page 28 and page 34, we have the following:

$$\begin{aligned} \mathrm{sn}\, U &= 0, \quad \text{where} \quad U = 2mK + 2inK' \\ \mathrm{cn}\, U &= 0, \quad \text{where} \quad U = (2m+1)K + 2inK' \\ \mathrm{dn}\, U &= 0, \quad \text{where} \quad U = (2m+1)K + i(2n+1)K' \end{aligned}$$

as well as

$$\mathrm{sn}\, a + \mathrm{sn}\, b = \frac{2\ \mathrm{sn}\left(\frac{a+b}{2}\right)\ \mathrm{cn}\left(\frac{a-b}{2}\right)\ \mathrm{dn}\left(\frac{a-b}{2}\right)}{1 - k^2 \mathrm{sn}^2\left(\frac{a+b}{2}\right) \mathrm{sn}^2\left(\frac{a-b}{2}\right)}$$

So, $\mathrm{sn}((v+u)/2) = 0$ or $\mathrm{cn}((v-u)/2) = 0$ or $\mathrm{dn}((v-u)/2) = 0$. Writing back $v = K + i\alpha K'$ and $u = -K + i\gamma K'$, we see that $\frac{v+u}{2} = i\frac{\alpha+\gamma}{2}$ and $\frac{v-u}{2} = K + i\frac{\alpha-\gamma}{2}$. Since we get no solution from either one of $\mathrm{sn}((v+u)/2) = 0$ or $\mathrm{dn}((v-u)/2) = 0$, we must have $(v-u)/2 \equiv (2m+1)K + 2inK'$, which forces us to choose $m = n = 0$ and $\gamma = \alpha$. This completes the proof of the claim.

Now we compare $\Phi(x)$ and $\Phi(-x)$:
$\Phi(x) = -i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(\frac{1}{2}(x+1/x))} = -i\, e^{\frac{\pi i}{2K}(K+i\alpha K')} = -i\,(i)\, e^{-\frac{\pi\alpha K'}{2K}} = e^{-\frac{\pi\alpha K'}{2K}}$

whereas

$\Phi(-x) = -i\, e^{\frac{\pi i}{2K}(-K+i\alpha K')} = -i\,(-i)\, e^{-\frac{\pi\alpha K'}{2K}} = -\, e^{-\frac{\pi\alpha K'}{2K}}$

proving that $\Phi(-z) = -\Phi(z)$ on the real line.

■

With $J(z) = \frac{1}{2}(z + \frac{1}{z})$, let us restate the definition of the extended Φ:

$$\Phi(z) = \begin{cases} -i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(z))}, & \text{if } z \text{ is in } \Omega^1 \\ i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(-z))}, & \text{if } z \text{ is in } \Omega^2 \end{cases}$$

or, equivalently speaking

$$\Phi(z) = \begin{cases} \overline{i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(-\bar{z}))}}, & \text{if } z \text{ is in } \Omega^1 \\ \overline{-i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(\bar{z}))}}, & \text{if } z \text{ is in } \Omega^2 \end{cases}$$

Now that Φ is a biholomorphism from $\Omega_r \cap \mathbb{D}$ onto $A_{\rho,1}$ *which fixes the unit circle*, it can be extended analytically to the remaining part of the Bell representative domain Ω_r beyond the unit circle $\mathbb{T}$ by the relation

$$\Phi(z) = \frac{1}{\overline{\Phi(\frac{1}{\bar{z}})}}$$

by yet another application of the Schwarz reflection principle. The final extended mapping Φ is now a biholomorphism from Ω_r onto $A_{\rho,1/\rho}$.

Observe that:

$w \in \Omega^1 \Leftrightarrow \frac{1}{\bar{w}} \in \Omega^3$

$w \in \Omega^2 \Leftrightarrow \frac{1}{\bar{w}} \in \Omega^4$

$w \in \Omega^3 \Leftrightarrow \frac{1}{\bar{w}} \in \Omega^1$

$w \in \Omega^4 \Leftrightarrow \frac{1}{\bar{w}} \in \Omega^2$

These relations help us explicitly write out the formula for the biholomorphism Φ throughout Ω_r as:

$$\Phi(z) = \begin{cases} -i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(z))} & \text{if } z \text{ is in } \Omega^1 \\ i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(-z))} & \text{if } z \text{ is in } \Omega^2 \\ -i\, e^{-\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(-z))} & \text{if } z \text{ is in } \Omega^3 \\ i\, e^{-\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(z))} & \text{if } z \text{ is in } \Omega^4 \end{cases}$$

or,

$$\Phi(z) = \begin{cases} \overline{i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(-\bar{z}))}} & \text{if } z \text{ is in } \Omega^1 \\ \overline{-i\, e^{\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(\bar{z}))}} & \text{if } z \text{ is in } \Omega^2 \\ \overline{i\, e^{-\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(\bar{z}))}} & \text{if } z \text{ is in } \Omega^3 \\ \overline{-i\, e^{-\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(-\bar{z}))}} & \text{if } z \text{ is in } \Omega^4 \end{cases}$$

Although it may seem peculiar, there is a good reason we want to write the formula for Φ as in the latter form above. The reason is that we will use both of these formulas throughout our computations, and in particular, in the computation of the Bergman kernel of the Bell representative domain.

From the formula representation of Φ, we can deduce relations that help us know how to take the conjugate of the composition of the elliptic integral with the Joukowski map $\mathrm{sn}^{-1} \circ J$. Let us go over this briefly.

Corollary 1

$$\overline{\mathrm{sn}^{-1}(J(z))} = -\mathrm{sn}^{-1}(-J(\bar{z})) \ \textit{if } z \in \Omega^1 \cup \Omega^4$$

as well as

$$\overline{\mathrm{sn}^{-1}(-J(z))} = -\mathrm{sn}^{-1}(J(\bar{z})) \ \textit{if } z \in \Omega^2 \cup \Omega^3$$

Proof Suppose without loss of generality that $z \in \Omega^1$. From the definition of the biholomorphism, we have

$$\overline{\exp\left(\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(-\bar{z}))\right)} = \exp\left(\frac{\pi i}{2K}\mathrm{sn}^{-1} J(z)\right)$$

from which we get

$$1 = e^{\frac{\pi i}{2K}[\mathrm{sn}^{-1}J(z) + \overline{\mathrm{sn}^{-1}(J(-\bar{z}))}]}$$

$\Rightarrow \ \mathrm{sn}^{-1}J(z) \equiv \overline{-\mathrm{sn}^{-1}(J(-\bar{z}))} \pmod{4nK}$ for some $n \in \mathbb{Z}$.

$\Rightarrow \overline{\mathrm{sn}^{-1}(J(z))} = -\mathrm{sn}^{-1}(-J(\bar{z}))$ if $z \in \Omega^1$.

The other relations are proved likewise. ∎

We finish the chapter by a consequence of the biholomorphism $\Phi : \Omega_{2/\sqrt{k}} \to A_{\rho,1/\rho}$ constructed above.

Theorem 2.2.1 *The Bell representative domain* $\Omega_{2/\sqrt{k}}$ *is biholomorphic to* $A_{\rho^2,1}$ *with* $\rho = e^{-\frac{\pi K'}{4K}}$. *Here* K *and* K' *are given by*

$$\begin{aligned} K &= \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \\ K' &= \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k'^2\xi^2)}} \end{aligned}$$

with $k^2 + k'^2 = 1$.

3. The Ahlfors map of Ω_r

Tegtmeyer [18] worked on the annulus $A_{\rho^2,1}$ and calculated the Ahlfors maps on this annulus at an arbitrary point. Since we have a conformal map $\Psi(z) = \rho\,\Phi(z)$ from Ω_r onto the annulus $A_{\rho^2,1}$, we can get all the Ahlfors maps of the Bell representative domain simply by composing these two maps and then multiplying by an appropriate constant of unit modulus.

Let us recall some of the results of Tegtmeyer.

Lemma 2 *If f_ρ is the Ahlfors map on the annulus $A_{\rho^2,1}$ with base point ρ, then*

$$f_\rho(z) = -i\,\sqrt{k}\,\mathrm{sn}\,(\frac{2Ki}{\pi}\log(\frac{z}{\rho}))$$

Using this result, we are actually able to write the Ahlfors map of an arbitrary point in the annulus $A_{\rho^2,1}$. Here is another result due to Tegtmeyer:

Lemma 3 *Suppose $w = r\,e^{i\gamma}$ is an arbitrary point in the annulus $A_{\rho^2,1}$. Then f_w, the Ahlfors map on this annulus with base point w, is given by*

$$f_w(z) = e^{i\gamma}\frac{f_\rho(e^{-i\gamma}z) - f_\rho(r)}{1 - f_\rho(e^{-i\gamma}z)\,f_\rho(r)}$$

Once we notice $f_\rho(r)$ is a real number between -1 and 1, we can see that this is a Möbius transformation of $f_\rho(e^{-i\gamma}z)$.

3.1 The general form of the Ahlfors maps of Ω_r

Composing f_w with Ψ will produce us proper holomorphic mappings from Ω_r onto the unit disk $\mathbb{D}$, and that is precisely how we get a grip of the Ahlfors maps of Ω_r.

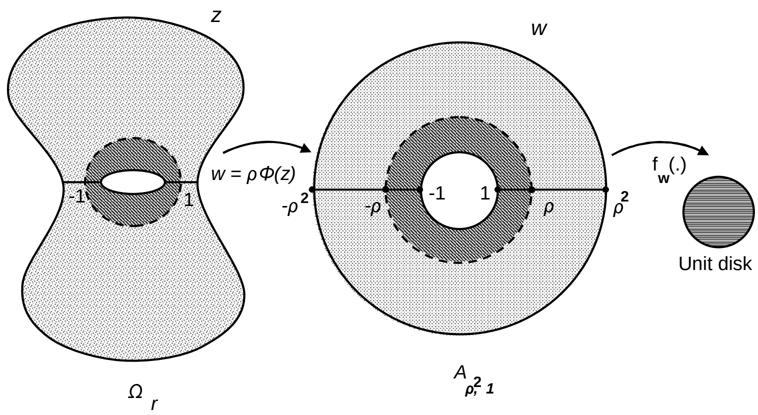

Figure 3.1. Ahlfors map of Ω_r can be found by composition.

Theorem 3.1.1 *The Ahlfors map of Ω_r with base point z_0 is given by*

$$g_{z_0}(z) \; = \; \lambda \, f_w \left(\rho \, \Phi(z) \right)$$

where $w \in A_{\rho^2,1}$ is such that $w = \rho\,\Phi(z_0)$ and $\lambda = \lambda(z_0)$ is a constant of unit modulus.

Proof Let $z_0 \in \Omega_r$ be given. Since f_w and $\Psi(.) = \rho\,\Phi(.)$ are holomorphic on their domains and f_w is two-to-one, $g_{z_0}(.)$ is a two-to-one holomorphic function from Ω_r onto $\mathbb{D}$. $g_{z_0}(.)$ is continuous on $\overline{\Omega}_r$ since so is f_w upto and on the boundary of the annulus. Furthermore, each boundary component of Ω_r is mapped 1-1 and onto the unit circle $\mathbb{T}$.

Note that $g_{z_0}(z_0) \; = \; \lambda \, f_w \left(\rho \, \Phi(z_0) \right) \; = \; \lambda \, f_w(w) = 0$.

Let us choose $\lambda = \frac{|\Phi'(z_0)|}{\Phi'(z_0)}$ and then we will get a positive derivative for g_{z_0} at the point z_0:

$g'_{z_0}(z_0) \; = \; \lambda \, f'_w \left(\rho \, \Phi(z_0) \right) \rho \, \Phi'(z_0) = f'_w(w) \, \rho \, |\Phi'(z_0)| \; > \; 0$

Therefore, $g_{z_0}(.)$ is a proper holomorphic map from the representative domain onto the unit disk $\mathbb{D}$ that maps $z_0 \mapsto 0$ and has positive derivative at z_0. Consequently, $g_{z_0}(.)$ is the Ahlfors map on Ω_r with base point z_0.

■

We will find explicit formulas for the Ahlfors map of Ω_r based at z_0 later.

The mapping $g_i(z) = \frac{1}{r}(z + \frac{1}{z})$ is an Ahlfors mapping of Ω_r with base point $z = i$. Although it is clear from the definition of Ω_r that $g_i(z)$ is a two-branched covering onto the unit disk with $g_i(i) = 0$ and $g_i'(i) = \frac{2}{r} > 0$ and hence is the Ahlfors map of Ω_r at $z = i$, let us reach this result of calculating it in an alternative way based on our recent observations.

Proposition 3.1.1 *The Ahlfors map of Ω_r with base point i is given by $f_{\rho i}(\rho\,\Phi(z))$, which is $\frac{1}{r}(z + \frac{1}{z})$.*

Proof It is easy to see that $\rho\,\Phi(i) = i\rho$. By the proposition above,

$$f_{\rho i}(\rho\,\Phi(z)) = i\, f_{\rho}(-i\,\rho\,\Phi(z)) = i\,(-i\sqrt{k})\,\mathrm{sn}\,(\frac{2Ki}{\pi}\log(-i\,\Phi(z)))$$

Now, let us simplify $\log(-i\,\Phi(z))$:

$$\log(-i\,\Phi(z)) = \begin{cases} \frac{\pi i}{2K}\mathrm{sn}^{-1}(J(z)) + \pi i & \text{if } z \text{ is in } \Omega^1 \\ \frac{\pi i}{2K}\mathrm{sn}^{-1}(-J(z)) & \text{if } z \text{ is in } \Omega^2 \\ -\frac{\pi i}{2K}\mathrm{sn}^{-1}(-J(z)) + \pi i & \text{if } z \text{ is in } \Omega^3 \\ -\frac{\pi i}{2K}\mathrm{sn}^{-1}(J(z)) & \text{if } z \text{ is in } \Omega^4 \end{cases}$$

so that

$$\frac{2Ki}{\pi}\log(-i\,\Phi(z)) = \begin{cases} -\mathrm{sn}^{-1}(J(z)) - 2K & \text{if } z \text{ is in } \Omega^1 \\ -\mathrm{sn}^{-1}(-J(z)) & \text{if } z \text{ is in } \Omega^2 \\ \mathrm{sn}^{-1}(-J(z)) + 2K & \text{if } z \text{ is in } \Omega^3 \\ \mathrm{sn}^{-1}(J(z)) & \text{if } z \text{ is in } \Omega^4 \end{cases}$$

so that $\mathrm{sn}(\frac{2Ki}{\pi}\log(-i\,\Phi(z))) = J(z) = \frac{1}{2}(z + \frac{1}{z})$ for any $z \in \Omega_r$.

It is easy to check that $\lambda = \frac{|\Phi'(i)|}{\Phi'(i)} = 1$ in this case so that

$$g_i(z) = \frac{\sqrt{k}}{2}(z + \frac{1}{z}) = \frac{1}{r}(z + \frac{1}{z}).$$

■

Here are a few simple calculations, since we will deal with the square root function $\sqrt{1 - J^2(z)}$, where by $\sqrt{\ }$, we mean the principal branch of the square root function:

$\sqrt{1-J^2(z)} = \sqrt{1-\frac{(z^2+1)^2}{4z^2}} = \sqrt{\frac{4z^2-(z^2+1)^2}{4z^2}} = \sqrt{-\frac{(z^2-1)^2}{4z^2}}$

Notice that $\sqrt{1-J^2(-i)} = \sqrt{1-0} = 1$ just like $\frac{i}{2}(\frac{(-i)^2-1}{-i}) = 1$. So, already having specified $\sqrt{1}=1$, we can actually *define* $\sqrt{1-J^2(z)} := \frac{i}{2}\frac{z^2-1}{z}$ for $z \in$ LHP.

Meanwhile, let us also consider $\sqrt{1\,-\,k^2\,J^2(z)}$: Due to the definition of Ω_r, the function $-k^2\,J^2(z)$ maps Ω_r onto $D_k(0)$, the disk of radius $k<1$ centered at the origin in a 4-to-1 fashion. Therefore, $1-k^2\,J^2(z)$ maps Ω_r onto $D_k(1)$, the disk of radius $k<1$ centered on the real axis at $z=1$. Since this is on the right-half plane, we can define a single-valued analytic branch of $\sqrt{1\,-\,k^2\,J^2(z)}$ by using the principal branch of the square root function.

Let us now calculate the Ahlfors mapping on Ω_r at the point $z=1$. Since $\Phi(1)=1$ and since $\Phi'(1)>0$ (so that $\lambda(1)=\frac{|\Phi'(1)|}{\Phi'(1)}=1$), this Ahlfors mapping is given by

$$g_1(z) = f_\rho(\rho\,\Phi(z)) = -i\sqrt{k}\,\mathrm{sn}(\frac{2Ki}{\pi}\log\Phi(z)).$$

If $z \in \Omega^1 \Rightarrow g_1(z) = -i\sqrt{k}\,\mathrm{sn}(\frac{2Ki}{\pi}[\frac{\pi i}{2K}\mathrm{sn}^{-1}J(z) - \frac{\pi i}{2}])$

$$\begin{aligned}
g_1(z) &= -i\sqrt{k}\ \mathrm{sn}(K-\mathrm{sn}^{-1}J(z)) \\
&= -i\sqrt{k}\ \frac{(\mathrm{sn}K)\ \mathrm{cn}(\mathrm{sn}^{-1}J(z))\ \mathrm{dn}(\mathrm{sn}^{-1}J(z))}{1-k^2\mathrm{sn}^2(\mathrm{sn}^{-1}J(z))} \\
&= -i\sqrt{k}\ \frac{\mathrm{cn}(\mathrm{sn}^{-1}J(z))}{\mathrm{dn}(\mathrm{sn}^{-1}J(z))} \\
&= -i\sqrt{k}\ \frac{\sqrt{1-J^2(z)}}{\sqrt{1-k^2J^2(z)}} \\
&= -i\sqrt{k}\ \frac{(i/2)(\frac{z^2-1}{z})}{\sqrt{1-k^2J^2(z)}} \quad \text{since } z \text{ is in the LHP} \\
g_1(z) &= \frac{\sqrt{k}}{2}\ \frac{z-\frac{1}{z}}{\sqrt{1-k^2J^2(z)}} \qquad (3.1)
\end{aligned}$$

Proposition 3.1.2 *The Ahlfors mapping on Ω_r with base point $z=1$ is given by* $g_1(z) = \frac{1}{r}\frac{z-\frac{1}{z}}{\sqrt{1-k^2J^2(z)}}$.

3.2 The location of the zeroes for the Ahlfors maps of Ω_r

We know that $g_{z_0}(z)$, the Ahlfors mapping with base point $z_0 \in \Omega_r$, is given by $\frac{|\Phi'(z_0)|}{\Phi'(z_0)} f_w(\,\rho\,\Phi(z)\,)$ where f_w is the Ahlfors mapping on the annulus $A_{\rho^2,1}$ with base point $w \in A_{\rho^2,1}$ and $w = \rho\,\Phi(z_0)$. Our aim is to locate the second nontrivial zero of the Ahlfors mapping $g_{z_0}(z)$, which trivially has a zero at $z = z_0$.

Theorem 3.2.1 $g_{z_0}(-\frac{1}{\overline{z_0}}) = 0$

Proof Let Ω be the Bell representative domain and let A denote the annulus $A_{\rho^2,1}$ for simplicity of notation. We already have the explicit biholomorphism Ψ from Ω onto A. Since $g_{z_0}(z) = \frac{S_\Omega(z,z_0)}{L_\Omega(z,z_0)}$ where S and L denote the Szegö and Garabedian kernel functions respectively, the claim is that the Szegö kernel function $S_\Omega(.,z_0)$ has its unique zero at $z = -\frac{1}{\overline{z_0}}$.

By the transformation formula for the Szegö kernels, we have

$$S_\Omega(z,w) = \sqrt{\Psi'(z)}\, S_A(\Psi(z),\Psi(w))\, \overline{\sqrt{\Psi'(w)}}$$

where we know by [18] that

$$S_A(z,w) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \frac{(z\bar{w})^n}{1+\rho^{4n+2}}$$

So we have

$$\begin{aligned}
S_\Omega(-\frac{1}{\overline{w}},w) &= \sqrt{\Psi'(-1/\overline{w})}\, S_A(\Psi(-\frac{1}{\overline{w}}),\Psi(w))\, \overline{\sqrt{\Psi'(w)}} \\
&= \frac{1}{2\pi} \sqrt{\Psi'(-1/\overline{w})}\, (\sum_{n=-\infty}^{\infty} \frac{[\Psi(-\frac{1}{\overline{w}})\overline{\Psi(w)}]^n}{1+\rho^{4n+2}})\, \overline{\sqrt{\Psi'(w)}} \\
&= \frac{1}{2\pi} \sqrt{\Psi'(-1/\overline{w})}\, (\sum_{n=-\infty}^{\infty} \frac{[\rho^2\,\Phi(-\frac{1}{\overline{w}})\overline{\Phi(w)}]^n}{1+\rho^{4n+2}})\, \overline{\sqrt{\Psi'(w)}} \\
&= \frac{1}{2\pi} \sqrt{\Psi'(-1/\overline{w})}\, (\sum_{n=-\infty}^{\infty} \frac{[-\rho^2\,\Phi(\frac{1}{\overline{w}})\overline{\Phi(w)}]^n}{1+\rho^{4n+2}})\, \overline{\sqrt{\Psi'(w)}} \\
&= \frac{1}{2\pi} \sqrt{\Psi'(-1/\overline{w})}\, (\sum_{n=-\infty}^{\infty} \frac{(-\rho^2)^n}{1+\rho^{4n+2}})\, \overline{\sqrt{\Psi'(w)}}
\end{aligned} \tag{3.2}$$

The sum in the middle reminds us of the way the second (nontrivial) zero of the Szegö kernel on the annulus $A_{\rho^2,1}$ is found as in [18]. We can rearrange the sum in any order we would like, due to the absolute convergence of the series.

$$\begin{aligned}
\sum_{n=-\infty}^{+\infty} \frac{(-\rho)^{2n}}{1+\rho^{4n+2}} &= \sum_{n=-\infty}^{-1} \frac{(-1)^n \rho^{2n}}{1+\rho^{4n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n}}{1+\rho^{4n+2}} \\
&= \sum_{j=1}^{\infty} \frac{(-1)^j \rho^{-2j}}{1+\rho^{-4j+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n}}{1+\rho^{4n+2}} \\
&= \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \rho^{-2j-2}}{1+\rho^{-4j-2}} \frac{\rho^{4j+2}}{\rho^{4j+2}} + \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n}}{1+\rho^{4n+2}} \\
&= \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \rho^{2j}}{\rho^{4j+2}+1} + \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n}}{1+\rho^{4n+2}} \\
&= 0 \qquad (3.3)
\end{aligned}$$

Consequently, the Ahlfors mapping $g_{z_0}(z)$ on Ω has a zero at $-1 / \overline{z_0}$.

Another Proof. We know that $g_{z_0}(z) = \lambda\, f_w(\rho\, \Phi(z))$ where $w = \rho\, \Phi(z_0)$. Since $\rho\, \Phi(-\frac{1}{\overline{z_0}}) = -\rho\, \Phi(\frac{1}{\overline{z_0}}) = -\rho\, \frac{1}{\overline{\Phi(z_0)}} = \frac{-\rho}{\overline{w/\rho}} = -\frac{\rho^2}{\bar{w}}$ and since by [18] we know $f_w(-\frac{\rho^2}{\bar{w}}) = 0$, we get the desired result that $g_{z_0}(-\frac{1}{\overline{z_0}}) = 0$.

■

3.3 The branch points for the Ahlfors maps of Ω_r

The branch points for the Ahlfors maps for the Bell representative domain are in close connection with those for the Ahlfors maps for the annulus, since

$$g'_{z_0}(z) = \lambda\, f'_w(\rho\Phi(z))\, \rho\, \Phi'(z)$$

where $w = re^{i\theta} = \rho\Phi(z_0)$ with $\rho^2 < r < 1$. We know from [18] that $f'_w(\pm i\rho e^{i\theta}) = 0$. So, if $z \in \Omega_r$ is such that $\Phi(z) = \pm i e^{i\theta} \in \mathbb{T}$, then $g'_{z_0}(z) = 0$. Clearly, the converse of this statement is also true. Due to the construction of Φ, we immediately see that the branch points lie on $\mathbb{T}$. Furthermore, if z and $\tilde{z}$ are the branch points of any $g_{z_0}(z)$, then $-z = \tilde{z}$.

Proposition 3.3.1 *Let $z_0 \in \Omega_r$ be arbitrary. The two branch points z_1, z_2 of the Ahlfors map $g_{z_0}(z)$ of Ω_r lie on the unit circle. Furthermore, $z_1 = -z_2$.*

3.4 The Ahlfors map of Ω_r at z_0 in explicit form

Here in this section we will determine explicit formulas for the Ahlfors map of Ω_r at z_0.

Lemma 4 *Let $z_0 \in \Omega_r$ be arbitrary. The Ahlfors maps g_{z_0} and $g_{\frac{-1}{\overline{z_0}}}$ satisfy*

$$g_{z_0}(z) \;=\; \beta\, g_{\frac{-1}{\overline{z_0}}}(z)$$

for some $|\beta| = 1$ for any $z \in \Omega_r$.

Proof We now already know that $g_{z_0}(z)$ and $g_{\frac{-1}{\overline{z_0}}}(z)$ have the same zeroes. z_0 and $-1/\overline{z_0}$ being the zeroes for $g_{z_0}(z)$ and $-1/\overline{z_0}$ and $-1/(\overline{-1/\overline{z_0}}) = z_0$ being the zeroes for $g_{\frac{-1}{\overline{z_0}}}(z)$. Consider the function $G(z) = \frac{g_{z_0}(z)}{g_{\frac{-1}{\overline{z_0}}}(z)}$. G is analytic inside Ω_r, and $|G| = 1$ on $\partial\Omega_r$, the boundary of Ω_r. By maximum modulus theorem, $|G| \leq 1$ in Ω_r. But, upon redefining $G(z)$ at z_0 and $-1/\bar{z}_0$ if necessary, G is zero-free in Ω_r, therefore, by maximum modulus theorem applied this time on $1/G$, we see that $|1/G| \leq 1$ in Ω_r. This implies that $|G| = 1$ in Ω_r and consequently, G is equal to a unimodular constant β in Ω_r. ■

Proposition 3.4.1 *The unimodular constant β defined in the lemma above is equal to*

$$\frac{|\Phi'(z_0)|}{|\Phi'(-1/\overline{z_0})|}\frac{\Phi'(-1/\overline{z_0})}{\Phi'(z_0)}.$$

Proof The constant β can be computed easily using the fact that

$$g_{z_0}(z) \;=\; \frac{|\Phi'(z_0)|}{\Phi'(z_0)}\, f_w\,(\rho\,\Phi(z)), \qquad g_{\frac{-1}{\overline{z_0}}}(z) \;=\; \frac{|\Phi'(-1/\overline{z_0})|}{\Phi'(-1/\overline{z_0})}\, f_{-\frac{\rho^2}{\overline{w}}}\,(\rho\,\Phi(z))$$

as well as $f_w(.) = f_{-\frac{\rho^2}{\overline{w}}}(.)$ [18]. Consequently, $\beta = \frac{|\Phi'(z_0)|}{|\Phi'(-1/\overline{z_0})|}\frac{\Phi'(-1/\overline{z_0})}{\Phi'(z_0)}$.

■

We know from geometric considerations that $z_0 \in \Omega^1 \iff -1/\overline{z_0} \in \Omega^4$, and likewise $z_0 \in \Omega^2 \iff -1/\overline{z_0} \in \Omega^3$. We will write down the formula for the Ahlfors map at $z_0 \in \Omega_j$ for $j = 1, 2$ and by the lemma above, it will cover the cases for $z_0 \in \Omega_j$ for $j = 3, 4$.

Suppose that $w = re^{i\gamma} = \rho\Phi(z_0)$. We know that we are able to write the Ahlfors map $g_{z_0}(z)$ as a unimodular multiple of $(f_w \circ \Psi)(z)$.

By Lemma 3.2,

$$f_w(z) = e^{i\gamma}\frac{f_\rho(e^{-i\gamma}z) - f_\rho(r)}{1 - f_\rho(e^{-i\gamma}z)\, f_\rho(r)}$$

which means

$$f_w(\rho\Phi(z)) = e^{i\gamma}\frac{f_\rho(e^{-i\gamma}\rho\Phi(z)) - f_\rho(r)}{1 - f_\rho(e^{-i\gamma}\rho\Phi(z))\, f_\rho(r)}$$

On the other hand, by Lemma 3.1, we have

$$\begin{aligned} f_\rho(e^{-i\gamma}\rho\Phi(z)) &= -i\sqrt{k}\,\mathrm{sn}(\frac{2Ki}{\pi}\log(e^{\frac{\pi i}{2K}(\mathrm{sn}^{-1}(J(z))-K)-i\gamma})) \\ &= -i\sqrt{k}\,\mathrm{sn}(\frac{2K\gamma}{\pi} + K - \mathrm{sn}^{-1}(J(z))) \qquad (3.4) \\ &= -i\sqrt{k}\,\mathrm{sn}(\frac{2K\gamma'}{\pi} - \mathrm{sn}^{-1}(J(z))), \text{ where } \gamma' = \gamma + \pi/2 \\ &= -i\sqrt{k}\,[\frac{\mathrm{sn}(\frac{2K\gamma'}{\pi})\frac{i}{2}\frac{z^2-1}{z}\sqrt{1-k^2J^2(z)} - J(z)\mathrm{cn}(\frac{2K\gamma'}{\pi})\mathrm{dn}(\frac{2K\gamma'}{\pi})}{1-k^2\,\mathrm{sn}^2(\frac{2K\gamma'}{\pi})J^2(z)}] \end{aligned}$$

(3.5)

Likewise

$f_\rho(r) = -i\sqrt{k}\,\mathrm{sn}(\frac{2Ki}{\pi}\log(\frac{r}{\rho}))$.

This means that once we have the expressions for γ, γ' and $\frac{r}{\rho}$ in terms of z_0, we will have the Ahlfors map at z_0 in terms of z_0 explicitly. Let us find the expressions for these variables in terms of z_0 for the the cases when $z_0 \in \Omega^1$ and $z_0 \in \Omega^2$. By one of the earlier remarks, this means we will be able to write down an arbitrary Ahlfors map on Ω_r.

Case 1. Suppose $z_0 \in \Omega^1$. Then

$$\begin{aligned} w\,\bar{w} = r^2 &= \rho\,\Phi(z_0)\,\overline{\rho\,\Phi(z_0)} \\ &= \rho^2\, e^{\frac{\pi i}{2K}[\,\mathrm{sn}^{-1}J(z_0)+\mathrm{sn}^{-1}J(-\overline{z_0})\,]}, \\ &= \rho^2\, e^{\frac{\pi i}{2K}[\,\mathrm{sn}^{-1}J(z_0)-\overline{\mathrm{sn}^{-1}J(z_0)}\,]}, \end{aligned} \tag{3.6}$$

which means

$$r/\rho = e^{\frac{\pi i}{4K}[\,\mathrm{sn}^{-1}J(z_0)+\mathrm{sn}^{-1}J(-\overline{z_0})\,]},$$

or, equivalently $r/\rho = e^{\frac{\pi i}{4K}[\,2i\,\mathrm{Im}(\mathrm{sn}^{-1}J(z_0))\,]} = e^{-\frac{\pi}{2K}[\,\mathrm{Im}(\mathrm{sn}^{-1}J(z_0))\,]}$.
Likewise, since $e^{i\gamma} = \frac{\rho}{r}\,\Phi(z_0)$

$$\begin{aligned} i\gamma &= \log(\frac{\rho}{r}\,\Phi(z_0)) \\ \gamma &= -i\,\log(e^{\frac{\pi i}{4K}[\,\mathrm{sn}^{-1}J(z_0)-\mathrm{sn}^{-1}J(-\overline{z_0})\,]}\,(-i)) \\ \gamma &= -i\,\log(e^{\frac{\pi i}{4K}[\,\mathrm{sn}^{-1}J(z_0)+\overline{\mathrm{sn}^{-1}J(z_0)}\,]}\,(-i)) \\ \gamma &= -i\,\frac{\pi i}{2K}\,[\mathrm{Re}(\mathrm{sn}^{-1}J(z_0))\,-\,K] \\ \gamma &= \frac{\pi}{2K}\,[\mathrm{Re}(\mathrm{sn}^{-1}J(z_0))\,-\,K] \\ \gamma' &= \frac{\pi}{2K}\,[\mathrm{Re}(\mathrm{sn}^{-1}J(z_0))] \end{aligned} \tag{3.7}$$

Therefore

$$e^{i\gamma} = e^{\frac{\pi i}{2K}[\mathrm{Re}\,\mathrm{sn}^{-1}J(z_0)-K]} \tag{3.8}$$

$$\begin{aligned} f_\rho(r) &= -i\sqrt{k}\,\mathrm{sn}(\frac{2Ki}{\pi}\log(e^{-\frac{\pi}{2K}[\,\mathrm{Im}(\mathrm{sn}^{-1}J(z_0))\,]})) \\ f_\rho(r) &= -i\sqrt{k}\,\mathrm{sn}\,(\frac{2Ki}{\pi}(-\frac{\pi}{2K}\,\mathrm{Im}(\mathrm{sn}^{-1}J(z_0))\,)) \\ f_\rho(r) &= +i\sqrt{k}\,\mathrm{sn}\,(i\,\mathrm{Im}(\mathrm{sn}^{-1}J(z_0))) \end{aligned} \tag{3.9}$$

$$f_\rho(e^{-i\gamma}\rho\Phi(z)) = -i\sqrt{k}\,[\frac{\mathrm{sn}(R(z_0))\frac{i}{2}\frac{z^2-1}{z}\sqrt{1-k^2J^2(z)} - J(z)\mathrm{cn}(R(z_0))\mathrm{dn}(R(z_0))}{1-k^2\,\mathrm{sn}^2(R(z_0))J^2(z)}] \tag{3.10}$$

where $R(z_0) = \mathrm{Re}(\mathrm{sn}^{-1}J(z_0))$

Proposition 3.4.2 *Suppose $z_0 \in \Omega^1$. The Ahlfors map $g_{z_0}(z)$ with base point z_0 is given by*

$$g_{z_0}(z) = \frac{|\Phi'(z_0)|}{\Phi'(z_0)}\, e^{\frac{\pi i}{2K}(R(z_0)-K)}\, \frac{A_{z_0}(z) - B_{z_0}(z)}{1 - A_{z_0}(z)B_{z_0}(z)}$$

where

$$\begin{aligned} A_{z_0}(z) &= -i\sqrt{k}\,[\frac{sn(R(z_0))\frac{i}{2}\frac{z^2-1}{z}\sqrt{1-k^2J^2(z)} - J(z)\,cn(R(z_0))\,dn(R(z_0))}{1-k^2\,sn^2(R(z_0))J^2(z)}] \\ B_{z_0}(z) &= +i\sqrt{k}\,sn\,(i\,I(z_0)) \end{aligned} \tag{3.11}$$

with $R(z_0) = Re(sn^{-1}J(z_0))$, and $I(z_0) = Im(sn^{-1}J(z_0))$

Now let us determine the Ahlfors maps in the second case.

Case 2. Suppose $z_0 \in \Omega^2$. Then

$$\begin{aligned} w\,\bar{w} = r^2 &= \rho\,\Phi(z_0)\,\overline{\rho\,\Phi(z_0)} \\ r^2 &= \rho^2\, e^{\frac{\pi i}{2K}[\,\mathrm{sn}^{-1}J(-z_0)+\mathrm{sn}^{-1}J(\overline{z_0})\,]} \\ \Rightarrow r/\rho &= e^{\frac{\pi i}{4K}[\,\mathrm{sn}^{-1}J(-z_0)+\mathrm{sn}^{-1}J(\overline{z_0})\,]} && (3.12) \\ \Rightarrow r/\rho &= e^{-\frac{\pi}{2K}[\mathrm{Im}\,\mathrm{sn}^{-1}J(-z_0)]} && (3.13) \end{aligned}$$

Likewise, since $e^{i\gamma} = \frac{\rho}{r}\,\Phi(z_0)$

$$\begin{aligned} i\gamma &= \log(\frac{\rho}{r}\,\Phi(z_0)) \\ \gamma &= -i\,\log(e^{\frac{\pi i}{4K}[\,\mathrm{sn}^{-1}J(-z_0)-\mathrm{sn}^{-1}J(\overline{z_0})\,]}\,(i)) \\ \gamma &= -i\,\log(e^{\frac{\pi i}{4K}[\,\mathrm{sn}^{-1}J(-z_0)+\overline{\mathrm{sn}^{-1}J(-z_0)}\,]}\,(i)) \\ \gamma &= -i\,\frac{\pi i}{2K}\,[\mathrm{Re}(\mathrm{sn}^{-1}J(-z_0))\,+\,K] \\ \gamma &= \frac{\pi}{2K}\,[\mathrm{Re}(\mathrm{sn}^{-1}J(-z_0))\,+\,K] \\ \Rightarrow \frac{2K\gamma}{\pi} + K &= \mathrm{Re}(\mathrm{sn}^{-1}J(-z_0)) + 2K \end{aligned} \tag{3.14}$$

Therefore

$$\begin{aligned}
e^{i\gamma} &= e^{\frac{\pi i}{2K}[\mathrm{Re}\,\mathrm{sn}^{-1}J(-z_0)+K]} && (3.15)\\
f_\rho(r) &= -i\sqrt{k}\,\mathrm{sn}(\frac{2Ki}{\pi}\log(e^{-\frac{\pi}{2K}[\mathrm{Im}(\mathrm{sn}^{-1}J(-z_0))]}))\\
f_\rho(r) &= -i\sqrt{k}\,\mathrm{sn}\,(\frac{2Ki}{\pi}(-\frac{\pi}{2K}\,\mathrm{Im}(\mathrm{sn}^{-1}J(-z_0))\,))\\
f_\rho(r) &= +i\sqrt{k}\,\mathrm{sn}\,(i\,\mathrm{Im}(\mathrm{sn}^{-1}J(-z_0))) && (3.16)\\
f_\rho(e^{-i\gamma}\rho\Phi(z)) &= +i\sqrt{k}\,[\frac{\mathrm{sn}(R_-(z_0))\frac{i}{2}\frac{z^2-1}{z}\sqrt{1-k^2J^2(z)}-J(z)\mathrm{cn}(R_-(z_0))\mathrm{dn}(R_-(z_0))}{1-k^2\,\mathrm{sn}^2(R_-(z_0))J^2(z)}] && (3.17)
\end{aligned}$$

where $R_-(z_0)=\mathrm{Re}(\mathrm{sn}^{-1}J(-z_0))$. This way we are able to write the Ahlfors map on Ω_r at any point $z_0\in\Omega^2$.

Proposition 3.4.3 *Suppose* $z_0\in\Omega^2$. *The Ahlfors map* $g_{z_0}(z)$ *with base point* z_0 *is given by*

$$g_{z_0}(z)=\frac{|\Phi'(z_0)|}{\Phi'(z_0)}\,e^{\frac{\pi i}{2K}(R_-(z_0)+K)}\,\frac{C_{z_0}(z)-D_{z_0}(z)}{1-C_{z_0}(z)D_{z_0}(z)}$$

where

$$\begin{aligned}
C_{z_0}(z) &= i\sqrt{k}\,[\frac{sn(R_-(z_0))\frac{i}{2}\frac{z^2-1}{z}\sqrt{1-k^2J^2(z)}-J(z)\,cn(R_-(z_0))\,dn(R_-(z_0))}{1-k^2\,sn^2(R_-(z_0))J^2(z)}]\\
D_{z_0}(z) &= +i\sqrt{k}\,sn\,(i\,I_-(z_0))
\end{aligned}$$

with $R_-(z_0)=Re(sn^{-1}J(-z_0))$, *and* $I_-(z_0)=Im(sn^{-1}J(-z_0))$

3.5 The Bell representative domain versus the annulus

We know *a priori* that the Bell representative domain has algebraic Bergman and Szegö kernel functions [4], while the annulus has non-algebraic ones [18]. Therefore, the Bell representative domain and the annulus diverge in nature in this regard. While that is certainly true, here are some of the things that the Bell representative domain Ω_r and the annulus $A_{\rho,1/\rho}$ have in common: Given z_0 in this annulus or Ω_r, they both have Ahlfors maps that have zeros at the very same location. Also, the two branch points of the Ahlfors map for either domain lie on the unit circle $\mathbb{T}$. Given the divergent (algebraic versus non-algebraic) natures of their classical domain functions, one may argue that the similarity between their Ahlfors maps suggests that Ω_r can be considered as the algebraic analogue of the annulus.

Theorem 3.5.1 *Suppose Ω_r and $A_{\rho,\frac{1}{\rho}}$ are biholomorphic, $a \in A_{\rho,\frac{1}{\rho}} \cap \Omega_r$ and consider the two Ahlfors maps with base point a of Ω_r and $A_{\rho,\frac{1}{\rho}}$, respectively.*

1. The zeros of each Ahlfors map are the same, given by $\{a, -1/\bar{a}\}$. In particular, the only zero of the Szegö kernel $S(z,a)$ of either domain is at $z = -1/\bar{a}$.

2. The branch points (i.e. the zeros of the derivative) of the Ahlfors map of either domain lie on the unit circle $\mathbb{T}$.

4. The Bergman kernel of Ω_r

We constructed a biholomorphism $\Phi : \Omega_{2/\sqrt{k}} \to A_{\rho,\frac{1}{\rho}}$. Let us define the map $\Psi : \Omega_{2/\sqrt{k}} \to A_{\rho^2,1}$ with $\Psi(z) = \rho\,\Phi(z)$ so that Ψ is also trivially a biholomorphism.

4.1 The Bergman kernel K_{Ω_r} and the Weierstrass function

We remember that $r = 2/\sqrt{k}$. By the transformation formula for the Bergman kernels, we have

$$K_{\Omega_r}(z,w) = \Psi'(z)\ K_{A_{\rho^2,1}}(\Psi(z),\Psi(w))\ \overline{\Psi'(w)} \tag{4.1}$$

There is a well-known identity [6,15] that represents the Bergman kernel of an annulus in terms of the Weierstrass $\wp$-function:

$$K_{A_{\rho^2,1}}(z,w) = \frac{1}{\pi z\bar{w}}\left\{\wp(\log z\bar{w};\omega_1,\omega_3) + \frac{\eta}{\omega_3} - \frac{1}{2\omega_1}\right\} \tag{4.2}$$

where $\wp$ is the Weierstrass $\wp$ function determined by the full periods $2\omega_1 = -4\log\rho$ and $2\omega_3 = 2\pi i$ here. Also $\eta = \zeta(i\pi;\ -2\log\rho, i\pi)$, the increment of the Weierstrass ζ-function for the imaginary period. Since $2\omega_1$ and $2\omega_3$ are the full periods of the $\wp$ function,

$$\wp(u;\omega_1,\omega_3) = \wp(u+2\omega_1;\omega_1,\omega_3) = \wp(u+2\omega_3;\omega_1,\omega_3).$$

Furthermore, it is convenient to introduce the increments e_1 and e_3 for the periods of the $\wp$-functions. Denote $\wp(\omega_1;\omega_1,\omega_3) = e_1$ and $\wp(\omega_3;\omega_1,\omega_3) = e_3$. It is also customary and convenient to denote $\omega_2 = -(\omega_1+\omega_3)$ and $\wp(\omega_2) = e_2$. Regardless of the real and imaginary periods of the Weierstrass function, it is an interesting fact that $e_1 + e_2 + e_3 = 0$ just as $\omega_1 + \omega_2 + \omega_3 = 0$.

We would like to simplify the right-hand side of equation (4.2) which could be written as,

$$K_{A_{\rho^2,1}}(\Psi(z),\Psi(w)) = \frac{1}{\pi\Psi(z)\overline{\Psi(w)}}\left\{\wp(\log(\Psi(z)\overline{\Psi(w)}); -2\log\rho, \pi i) + C\right\}$$

where $C = C(k)$ is a real constant (since $\frac{\eta}{\pi i}$ is a real number [12]). We need to simplify $\wp(\log(\Psi(z)\overline{\Psi(w)}))$. We claim that this is an algebraic function in z and w.

Suppose we pick $z \in \Omega^1$ and $w \in \Omega^2$. Notice that

$$\begin{aligned} \log(\Psi(z)\overline{\Psi(w)}) &= \log\left\{ (-\rho^2)\, \mathrm{e}^{\frac{\pi i}{2K}[\mathrm{sn}^{-1}J(z)+\mathrm{sn}^{-1}J(\overline{w})]}\right\} \\ &= \log\left\{ \mathrm{e}^{\frac{\pi i}{2K}[\mathrm{sn}^{-1}J(z)+\mathrm{sn}^{-1}J(\overline{w})-\frac{4Ki}{\pi}\log\rho+2K]}\right\} \\ &= \frac{\pi i}{2K}[\mathrm{sn}^{-1}J(z)+\mathrm{sn}^{-1}J(\overline{w}) - \frac{4Ki}{\pi}\log\rho + 2K] + 2N\pi i \end{aligned}$$

At this point, we recall the homogeneity property of the $\wp$-function. It is a standard result in elliptic functions that the Weierstrass $\wp$-function enjoys [8, 12]

$$\wp(\lambda z; \lambda\omega_1, \lambda\omega_3) = \lambda^{-2}\wp(z; \omega_1, \omega_3). \tag{4.3}$$

With the choice of $\lambda = \frac{i\pi}{2K}$, we obtain

$$\begin{aligned} \wp(\log(\Psi(z)\overline{\Psi(w)})) &= \wp(\frac{\pi i}{2K}[\mathrm{sn}^{-1}J(z)+\mathrm{sn}^{-1}J(\overline{w}) - \frac{4Ki}{\pi}\log\rho + 2K] + 2N\pi i\,;\, -2\log\rho, \pi i) \\ &= \wp(\frac{\pi i}{2K}[\mathrm{sn}^{-1}J(z)+\mathrm{sn}^{-1}J(\overline{w}) - \frac{4Ki}{\pi}\log\rho + 2K]; -2\log\rho, \pi i) \\ &= -\frac{4K^2}{\pi^2}\,\wp(\mathrm{sn}^{-1}J(z)+\mathrm{sn}^{-1}J(\overline{w}) - \frac{4Ki}{\pi}\log\rho + 2K; 2K, -\frac{4Ki}{\pi}\log\rho) \\ &= -\frac{4K^2}{\pi^2}\,\wp(\mathrm{sn}^{-1}J(z)+\mathrm{sn}^{-1}J(\overline{w}) + 2K + iK'; 2K, iK') \end{aligned} \tag{4.4}$$

where the last step comes from the fact that $-4Ki\log\rho/\pi = iK'$, since we chose ρ as $\rho = e^{-\pi K'/4K}$.

The Weierstrass $\wp$-function and the Jacobi elliptic function are related through certain formulas, depending on how the half-periods ω_1 and ω_3 of the $\wp$-function connect with the so-called "quarter periods" K and K' of the Jacobi elliptic functions. A number of important formulas and relations that connect the two functions are obtained in the special case when $\omega_1 = K$ and $\omega_3 = iK'$. The relation in this case is given by

$$\wp(u; K, iK') = \frac{1}{\mathrm{sn}^2(u)} - \frac{1}{3}(1 + k^2). \tag{4.5}$$

Furthermore, in this case, $e_1 = \wp(K; K, iK') = \frac{1}{3}(2-k^2)$, $e_2 = \wp(-K-iK'; K, iK') = \frac{1}{3}(2k^2 - 1)$ and $e_3 = \wp(iK'; K, iK') = -\frac{1}{3}(1 + k^2)$ (see [8, 12]).

In what we had earlier, our real half-period is not K but $2K$. Since it will make our computations much easier and we would like to use equation 4.5, we use what is called a modular transformation on the real period of $\wp(u; 2K, iK')$. The following [12] is a good application of elementary complex analysis on the $\wp$-function.

Lemma 5 *The $\wp$ function has the modular transformation property that*

$$4\wp(u; 2\omega_1, \omega_3) = \wp(\frac{1}{2}u; \omega_1, \omega_3) + \wp(\frac{1}{2}u + \omega_3; \omega_1, \omega_3) - e_3.$$

Proof We consider the elliptic function $f(u) = \wp(\frac{1}{2}u; \omega_1, \omega_3) + \wp(\frac{1}{2}u + \omega_3; \omega_1, \omega_3)$. Because of the 1/2 factor, f is periodic with full-periods $4\omega_1$ and $2\omega_3$ and has a double pole at the origin (and at every point in the complex plane of the form $4m\omega_1 + 2n\omega_3$). The singular part of $f(u)$ around the origin is $(u/2)^{-2} = 4/u^2$. Since the same is true for the elliptic function $4\wp(u; 2\omega_1, \omega_3)$, by Liouville's theorem, $4\wp(u; 2\omega_1, \omega_3) - f(u) \equiv$ constant in $\mathbb{C}$, since it is holomorphic and bounded in any fundamental rectangle and hence in $\mathbb{C}$. One can put $u = \omega_3$ or let $u \to 0$ to conclude that the constant above is indeed $e_3 = \wp(\omega_3; \omega_1, \omega_3)$. ■

4.2 Computing K_{Ω_r}

For simplicity of notation, introduce $H(.) = (\mathrm{sn}^{-1} \circ J)(.)$. We would like to simplify $4\wp(\mathrm{sn}^{-1}J(z) + \mathrm{sn}^{-1}J(\overline{w}) + 2K + iK'\,; 2K, iK')$ ignoring the constant $-\frac{K^2}{\pi^2}$ in front of the Weierstrass function for now. By the lemma above, we have

$$
\begin{aligned}
4\,\wp(H(z) + H(\overline{w}) + 2K + iK';\, 2K, iK') &= \wp(\frac{1}{2}(H(z) + H(\overline{w})) + K + \frac{iK'}{2};\, K, iK') \\
&\quad + \wp(\frac{1}{2}(H(z) + H(\overline{w})) + K + \frac{3iK'}{2};\, K, iK') \\
&- \wp(iK';\, K, iK')
\end{aligned}
\tag{4.6}
$$

Let us simplify the right-hand side:

$$
\begin{aligned}
\ldots\ldots &= \frac{1}{sn^2(\frac{1}{2}(H(z) + H(\overline{w})) + K + \frac{iK'}{2})} \\
&\quad + \frac{1}{sn^2(\frac{1}{2}(H(z) + H(\overline{w})) + K + \frac{3iK'}{2})} - \frac{1}{3}(1 + k^2) \\
&= \frac{1 + dn(H(z) + H(\overline{w}) + 2K + iK')}{1 - cn(H(z) + H(\overline{w}) + 2K + iK')} \\
&\quad + \frac{1 + dn(H(z) + H(\overline{w}) + 2K + 3iK')}{1 - cn(H(z) + H(\overline{w}) + 2K + 3iK')} - \frac{1}{3}(1 + k^2) \\
&= \frac{1 + dn(H(z) + H(\overline{w}) + 2K + iK')}{1 - cn(H(z) + H(\overline{w}) + 2K + iK')} \\
&\quad + \frac{1 - dn(H(z) + H(\overline{w}) + 2K + iK')}{1 + cn(H(z) + H(\overline{w}) + 2K + iK')} - \frac{1}{3}(1 + k^2)
\end{aligned}
\tag{4.7}
$$

by the half-angle formula for the elliptic sine sn and since

$$dn(u + 2iK') = -dn(u),\;\; cn(u + 2iK') = -cn(u).$$

The previous equality then becomes

$$
\begin{aligned}
&= \frac{2 + 2\, dn(H(z)+H(\overline{w})+2K+iK')\, cn(H(z)+H(\overline{w})+2K+iK')}{sn^2(H(z)+H(\overline{w})+2K+iK')} - \frac{1}{3}(1+k^2) \\
&= \frac{2 \,+\, 2(-i\, cs(H(z)+H(\overline{w})))(-\frac{1}{ik}\, ds(H(z)+H(\overline{w})))}{1\,/\,k^2\, sn^2(H(z)+H(\overline{w}))} - \frac{1}{3}(1+k^2) \\
&= \left\{2 + \frac{2}{k}\frac{cn(H(z)+H(\overline{w}))}{sn(H(z)+H(\overline{w}))}\frac{dn(H(z)+H(\overline{w}))}{sn(H(z)+H(\overline{w}))}\right\} k^2\, sn^2(H(z)+H(\overline{w})) - \frac{1}{3}(1+k^2) \\
&= k\left\{2k\, sn^2(H(z)+H(\overline{w})) \,+\, 2\, cn(H(z)+H(\overline{w}))\, dn(H(z)+H(\overline{w}))\right\} - \frac{1}{3}(1+k^2)
\end{aligned}
\tag{4.8}
$$

Let us now call $\tilde{S}(z,w) = sn^2(H(z)+H(\overline{w}))$, as well as $\tilde{C}(z,w) = cn(H(z)+H(\overline{w}))$ and $\tilde{D}(z,w) = dn(H(z)+H(\overline{w}))$ and simplify these using the addition formulas for the Jacobi elliptic functions:

$$
\begin{aligned}
\tilde{S}(z,w) &= \left\{\frac{J(z)\sqrt{1-J^2(\overline{w})}\sqrt{1-k^2J^2(\overline{w})} + J(\overline{w})\sqrt{1-J^2(z)}\sqrt{1-k^2J^2(z)}}{1 \,-\, k^2\, J^2(z)J^2(\overline{w})}\right\}^2 \\
&= \left\{\frac{J(z)(\frac{i}{2}\frac{\bar{w}^2-1}{\bar{w}})\sqrt{1-k^2J^2(\overline{w})} + J(\overline{w})(\frac{i}{2}\frac{z^2-1}{z})\sqrt{1-k^2J^2(z)}}{1 \,-\, k^2\, J^2(z)J^2(\overline{w})}\right\}^2 \\
&= -\left\{\frac{J(z)(\frac{\bar{w}^2-1}{2\bar{w}})\sqrt{1-k^2J^2(\overline{w})} + J(\overline{w})(\frac{z^2-1}{2z})\sqrt{1-k^2J^2(z)}}{1 \,-\, k^2\, J^2(z)J^2(\overline{w})}\right\}^2
\end{aligned}
\tag{4.9}
$$

Likewise,

$$
\begin{aligned}
\tilde{C}(z,w) &= \frac{\sqrt{1-J^2(z)}\sqrt{1-J^2(\overline{w})} \,-\, J(z)J(\bar{w})\sqrt{1-k^2J^2(z)}\sqrt{1-k^2J^2(\overline{w})}}{1 \,-\, k^2\, J^2(z)J^2(\overline{w})} \\
&= \frac{(\frac{i}{2}\frac{z^2-1}{z})(\frac{i}{2}\frac{\bar{w}^2-1}{\bar{w}}) \,-\, J(z)J(\bar{w})\sqrt{1-k^2J^2(z)}\sqrt{1-k^2J^2(\overline{w})}}{1 \,-\, k^2\, J^2(z)J^2(\overline{w})} \\
&= \frac{-\frac{1}{4}(\frac{z^2-1}{z})(\frac{\bar{w}^2-1}{\bar{w}}) \,-\, J(z)J(\bar{w})\sqrt{1-k^2J^2(z)}\sqrt{1-k^2J^2(\overline{w})}}{1 \,-\, k^2\, J^2(z)J^2(\overline{w})}
\end{aligned}
\tag{4.10}
$$

and, finally

$$\begin{aligned}
\tilde{D}(z,w) &= \frac{\sqrt{1-k^2J^2(z)}\sqrt{1-k^2J^2(\overline{w})}\ -\ k^2J(z)J(\bar{w})\sqrt{1-J^2(z)}\sqrt{1-J^2(\overline{w})}}{1\ -\ k^2\ J^2(z)J^2(\overline{w})} \\
&= \frac{\sqrt{1-k^2J^2(z)}\sqrt{1-k^2J^2(\overline{w})}\ -\ k^2J(z)J(\bar{w})(\frac{i}{2}\frac{z^2-1}{z})(\frac{i}{2}\frac{\bar{w}^2-1}{\bar{w}})}{1\ -\ k^2\ J^2(z)J^2(\overline{w})} \\
&= \frac{\sqrt{1-k^2J^2(z)}\sqrt{1-k^2J^2(\overline{w})}\ +\ \frac{k^2}{4}J(z)J(\bar{w})(\frac{z^2-1}{z})(\frac{\bar{w}^2-1}{\bar{w}})}{1\ -\ k^2\ J^2(z)J^2(\overline{w})} \qquad (4.11)
\end{aligned}$$

So we get that

$$\wp(\log(\Psi(z)\overline{\Psi(w)})) = -\frac{K^2}{\pi^2}\left\{2k^2\tilde{S} + k\tilde{C}\tilde{D} - \frac{1}{3}(1+k^2)\right\}$$

with $\tilde{S}, \tilde{C}, \tilde{D}$ as above. On the other hand, we know that for $z \in \Omega^1, w \in \Omega^2$:

$$\begin{aligned}
\Psi(z) &= -\rho i\, e^{\frac{\pi i}{2K} sn^{-1} J(z)} \\
\Rightarrow \Psi'(z) &= \Psi(z)\,(\frac{\pi i}{2K})\frac{J'(z)}{\sqrt{1-J^2(z)}\sqrt{1-k^2J^2(z)}} \\
\Rightarrow \Psi'(z) &= \Psi(z)\,(\frac{\pi i}{2K})\frac{\frac{1}{2}(\frac{z^2-1}{z^2})}{\frac{i}{2}(\frac{z^2-1}{z})\sqrt{1-k^2J^2(z)}} \\
\Rightarrow \frac{\Psi'(z)}{\Psi(z)} &= \frac{\pi}{2K\, z\,\sqrt{1-k^2J^2(z)}} \qquad (4.12)
\end{aligned}$$

and

$$\begin{aligned}
\Psi(w) &= \rho i\, e^{\frac{\pi i}{2K} sn^{-1} J(-w)} \\
\Rightarrow \Psi'(w) &= \Psi(w)\,(\frac{\pi i}{2K})\frac{J'(-w)(-1)}{\sqrt{1-J^2(-w)}\sqrt{1-k^2J^2(-w)}} \\
\Rightarrow \Psi'(w) &= \Psi(w)\,(\frac{\pi i}{2K})\frac{\frac{1}{2}(\frac{w^2-1}{w^2})(-1)}{-\frac{i}{2}(\frac{w^2-1}{w})\sqrt{1-k^2J^2(w)}} \\
\Rightarrow \frac{\Psi'(w)}{\Psi(w)} &= \frac{\pi}{2K\, w\,\sqrt{1-k^2J^2(w)}} \\
\Rightarrow \overline{(\frac{\Psi'(w)}{\Psi(w)})} &= \frac{\pi}{2K\, \bar{w}\,\sqrt{1-k^2J^2(\bar{w})}} \qquad (4.13)
\end{aligned}$$

due to the fact that $\overline{\sqrt{1-k^2J^2(z)}}$ and $\sqrt{1-k^2J^2(\bar{z})}$ agree on the real axis in Ω_r. Consequently, the Bergman kernel function $K_{\Omega_r}(z,w)$ for Ω_r becomes

$$K_{\Omega_r}(z,w) = \Psi'(z)\frac{1}{\pi\Psi(z)\overline{\Psi(w)}}\left\{\wp(\log(\Psi(z)\overline{\Psi(w)}); -2\log\rho, \pi i) + C(k)\right\}\overline{\Psi'(w)}$$

$$K_{\Omega_r}(z,w) = \frac{\pi\left\{(2k^2\tilde{S}(z,w) + k\tilde{C}(z,w)\tilde{D}(z,w) - \frac{1}{3}(1+k^2))(-\frac{K^2}{\pi^2}) + C(k)\right\}}{4K^2\, z\,\bar{w}\,\sqrt{1-k^2J^2(z)}\,\sqrt{1-k^2J^2(\bar{w})}} \tag{4.14}$$

which is an algebraic and holomorphic function in z and $\bar{w}$ and satisfies the Hermitian property. The same formula is obtained for each choice of z in any one of Ω^2, Ω^3 or Ω^4 and likewise for w in any one of Ω^1, Ω^3 or Ω^4 in similar way such as above, however that is not necessary. The formula for the Bergman kernel function is true throughout all of Ω_r simply by analytic continuation.

4.3 Remarks on the Bergman kernel

Computing the Bergman kernel for a domain in the plane is very hard, if not impossible, except for simply connected domains. Thus, discovering the explicit formula for the Bergman kernel of Ω_r is remarkable. Aside from that, the discovery has a wider significance and general implications.

From a theoretical point of view, by the transformation formula in the beginning of the chapter, the Bergman kernel of any 2-connected planar domain can be computed as long as a biholomorphic map from that domain onto Ω_r for some $r > 2$ is found. This means that for doubly-connected domains in the plane, the Bell representative domain takes on the role of the unit disk in the simply connected case. In other words, it is a candidate for playing the role of the unit disk in the doubly-connected case.

Furthermore, we notice that the degree of the algebraic function $K_{\Omega_r}(z,w)$ above is independent of the parameter k, and hence independent of $r = 2/\sqrt{k}$. Therefore, from a practical point of view, the degree of the algebraic Bergman kernel function of our representative domain is seen to be independent of the parameter r.

We conclude by tying this to an earlier well-known fact: Every two-connected domain is biholomorphic to a two-connected Bell representative domain with an algebraic Bergman kernel function whose degree is independent of the modulus of that domain.

5. Future research

Here is a glimpse of problems that are yet to be explored and thoroughly worked on.

Green's function for Ω_r. The Green's function $G(z,w)$ and the Bergman kernel function $K(z,w)$ are related through the equation [2]

$$K(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \bar{w}}$$

Now that the Bergman kernel for our representative domain is found, we would like to use it to compute the Green's function. This may also have consequences in other fields, such as solving PDEs as well as in complex analysis.

Proper holomorphic mappings $f : \Omega_r \to \mathbb{D}$. We would like to determine the precise form of the proper holomorphic mappings from Ω_r into the unit disk $\mathbb{D}$. Tegtmeyer [18] determined the proper holomorphic mappings from the annulus $A_{\rho^2,1}$ into $\mathbb{D}$ in terms of the Ahlfors maps of the annulus. This will help us figure out perhaps other similarities and differences between Ω_r and $A_{\rho,1/\rho}$.

Carathéodory metric for Ω_r. Can we compute the Carathéodory metric for the Bell representative domain? How about the infinitesimal Carathéodory distance? Ahlfors map is the extremal function for the Carathéodory metric [5]. The doubly connected planar domains on which this metric is computable are rare [11, 14, 16]. The Bell representative domain is a candidate of a domain for which this can be done. We plan to compute the infinitesimal Carathéodory metric via a special pair of Ahlfors maps as explained in [4]. In Chapter Three, we found a pair of Ahlfors maps that do not share zeros. This should help us calculate the infinitesimal Carathéodory metric for Ω_r.

Degree in the annulus. For any n-connected domain Ω in the plane, according to the results in [3],

$$K_\Omega(z,w) = f_a'(z)\,\overline{f_a'(w)}\,\mathcal{R}(f_a(z), f_b(z), \overline{f_a(w)}, \overline{f_b(w)})$$

where $\mathcal{R}$ is a rational function of the four complex variables and where f_a and f_b are any pair of Ahlfors maps with $f_a(b) \neq 0$ as explained in [3].

We would like to find out what the relationship between the degree of the rational function and the modulus of the domain is, and in particular how this comes into play for the annulus. In other words, we would like to find how the degree of the rational function $\mathcal{R}$ above depends on the modulus of the annulus.

Szegö kernel of Ω_r. Szegö kernel of the Bell representative domain *a priori* is shown to be algebraic [4], although no closed formula for it was given there. The Szegö kernel of the annulus is a non-algebraic function, however since we have a biholomorphism between the Bell representative domain and an annulus, we can use our approach in Chapter Four to explicitly calculate it via the transformation formula between the kernels of these domains.

LIST OF REFERENCES

LIST OF REFERENCES

[1] N. Aronszajn, *Theory of Reproducing Kernels*, Transactions of the American Mathematical Society, Vol. 68, No. 3 (1950), pp. 337-404.

[2] S. Bell, *The Cauchy transform, potential theory, and conformal mapping*, CRC Press, Boca Raton, (1992).

[3] S. Bell, *Ahlfors maps, the double of a domain, and complexity in potential theory and conformal mapping*, Journal d'Analyse Mathematique 78 (1999), pp. 329-344.

[4] S. Bell, *Finitely generated function fields and complexity in potential theory in the plane*, Duke Mathematical Journal 98 (1999), pp. 187-207.

[5] S. Bell, *Möbius transformations, the Carathéodory metric, and the objects of complex analysis and potential theory in multiply connected domains*, Michigan Math. J. 51 (2003), pp. 351-362.

[6] S. Bergman, *The Kernel Function*, American Mathematical Society, (1950).

[7] D. G. Crowdy, *Quadrature domains and fluid dynamics*, in "Quadrature domains and applications", Harold Shapiro Anniversary volume, Birkhauser, (2005).

[8] H. Hancock, *Lectures on the Theory of Elliptic Functions*, Wiley, New York, (1910).

[9] M. Jeong, J. Oh, and M. Taniguchi, *Equivalence problem for annuli and Bell representations in the plane*, J. Math. Anal. Appl., Vol. 325, No. 2 (2007), pp. 1295-1305.

[10] S. Kanas and T. Sugawa, *On Conformal Representations of the Interior of an Ellipse* Annales Academiae Scientiarum Fen., vol 31 (2006), pp. 329-348.

[11] S. G. Krantz, *The Carathéodory and Kobayashi Metrics and Applications in Complex Analysis*, arxiv:math.CV/0608772v1 , Preprint (2006).

[12] D. Lawden, *Elliptic Functions and Applications*, Springer-Verlag, (1989).

[13] Z. Nehari, *Conformal Mapping*, Dover Publications, (1950).

[14] G. Schmieder, *On Schwarz's Lemma in Multiply Connected Domains*, CMFT 5 No.1 (2005), pp. 159-184.

[15] A. Sebbar, T. Falliero, *Equilibrium Point of Green's Function for the Annulus and the Eisenstein Series*, AMS (2007).

[16] R. R. Simha, *The Carathéodory metric of the annulus*, Proc. AMS 50 (1975), pp. 162-166.

[17] G. Szegö, *Conformal mapping of the interior of an ellipse onto a circle*, Amer. Math. Monthly 57 (1950), pp. 474-478.

[18] T. Tegtmeyer, *The Ahlfors map and Szegö kernel in multiply connected domains*, Ph.D. Thesis, Purdue University, West Lafayette, (1998).

VITA

VITA

Mustafa Ersin Değer was born on August 20, 1974, in the city of Van, Turkey. He graduated from Kadıköy Anadolu Lisesi, İstanbul in June of 1992. He received his Bachelor of Science in Mathematics from Bilkent University, Ankara, Turkey in June of 1996. He received his Master of Science in Applied Mathematics from the University of Minnesota, Minneapolis in 2000. He received his PhD in Mathematics from Purdue University in August, 2007.