GENERALIZED BUNCE–DEDDENS ALGEBRAS

STEFANOS ORFANOS

Abstract. We define a broad class of crossed product C*-algebras of the form $C(\tilde{G}) \rtimes G$, where G is a discrete countable amenable residually finite group, and $\tilde{G}$ is a profinite completion of G. We show that they are unital separable simple nuclear quasidiagonal C*-algebras, of real rank zero, stable rank one, with comparability of projections and with a unique trace.

Introduction

Our object of study will be a family of crossed products, which we call the *generalized Bunce–Deddens algebras*, because their construction generalizes a well-known construction of the classical Bunce–Deddens algebras, with $\mathbb{Z}$ being replaced by any discrete countable amenable residually finite group G, and the odometer action appropriately generalized. Such actions have been considered in the literature, in topological dynamics and quite recently in von Neumann algebra theory, but the corresponding crossed product C*-algebras have not been explicitly described. We should mention here that the generalized Bunce–Deddens algebras appearing in this paper are different from the ones introduced by D. Kribs in [5].

The generalized Bunce–Deddens algebras turn out to have many desirable properties, being simple nuclear quasidiagonal, and having real rank zero, stable rank one, a unique trace and comparability of projections. The ultimate goal of classifying such algebras is not realized in this paper; however, it is conjectured that they will have tracial rank zero or finite decomposition rank. Another open problem is to compute their ordered K–theory (possibly, their Elliott invariant), which may be within reach for certain groups.

1. Amenable residually finite groups and profinite completions

We start with the definition of amenability. Throughout this paper, G will be assumed to be discrete and countable.

Definition 1. A group G is *amenable* if there is a sequence $e \in F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n \ldots$ of finite sets of G such that:

$$\bigcup_{n \geq 1} F_n = G \text{ and } \lim_{n \to \infty} \frac{|F_n \triangle F_n s|}{|F_n|} = 0 \text{ for all } s \in G$$

Every amenable group has also left Følner sequences, i.e., sequences $(F_n)_n$ that also exhaust the group G and satisfy

$$\lim_{n \to \infty} \frac{|F_n \triangle s F_n|}{|F_n|} = 0 \text{ for all } s \in G$$

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as well as sequences that are both left and right Følner. Unless noted otherwise, a *Følner sequence* will be a right Følner sequence from now on.

Definition 2. The group G is *residually finite* if it has a separating family of finite index normal subgroups.

In other words, for every finite set F in G, there is a normal subgroup L of finite index in G, such that the quotient map $G \to G/L$ is injective, when restricted to F.

A *tiling* of G is a decomposition $G = KL$, where K is called the *tile* and L is the set of *tiling centers*, so that every $x \in G$ is uniquely written as a product of an element in K and an element in L. Tilings are an important feature of amenable residually finite groups, as we will see below.

Lemma 1. *Suppose G is an amenable and residually finite group, and assume a separating nested sequence of finite index subgroups L_n of G is given. Then there exists a Følner sequence $(F_n)_n$ and a sequence of finite subsets $K_n \supset F_n$ such that G has a tiling of the form $G = K_n L_n$ for all $n \geq 1$.*

The proof is based on the previous definitions. We will need a stronger result concerning the asymptotic behavior of tiles for G. The following theorem, essentially due to B. Weiss from [11], constructs *Følner tiles* for every separating nested sequence of finite index normal subgroups of G. The statement below comes from [3].

Theorem 2 (Weiss)**.** *For every discrete countable amenable residually finite group G and every separating nested sequence of finite index normal subgroups $(L_n)_n$, there is a sequence of sets $(K_n)_n$ that is left and right Følner, such that $G = K_n L_n$ is a tiling for all $n \geq 1$.*

Proofs of this result can be found in [3] or [11].

In what follows, we refine the above result to get left/right Følner sets that tile, not only the group G, but also subsequent tiles. First, we recall an equivalent formulation of Følner's characterization of amenability.

Definition 3. The *S-boundary* of a set K, denoted $\partial_S K$, is the set $SK \cap SK^c = \{x \in G : S^{-1}x \cap K \neq \emptyset \text{ and } S^{-1}x \cap K^c \neq \emptyset\}$.

Lemma 3. *The left Følner condition for K_n can be written:*

$$\frac{|\partial_S K_n|}{|K_n|} \to 0 \text{ as } n \to \infty \text{ for any fixed finite } S \subset G.$$

Indeed, $SK_n = (SK_n \cap SK_n^c) \cup (SK_n \setminus SK_n^c) = (SK_n \cap SK_n^c) \cup (\bigcap_{s \in S} sK_n)$, and both $\frac{|SK_n|}{|K_n|}$, $\frac{|\cap_{s\in S} sK_n|}{|K_n|}$ converge to 1 by Theorem 16.16 of [8].

The improved tiling result is the following:

Theorem 4. *If G is a discrete countable amenable residually finite group G and $(L_n)_n$ is a separating nested sequence of finite index normal subgroups of G, then there exist finite subsets K_n of G and integers $n_1 < n_2 < \cdots$ such that:*

(1) *$(K_n)_n$ is a left and right Følner sequence, and $G = K_n L_n$ is a tiling for all $n \geq 1$*

(2) *$K_{n_{k+1}}$ is the disjoint union of right translates of K_{n_k}, for all $k = 1, 2, \ldots$.*

Proof. The first property comes from Weiss's theorem. To prove the second property, let $\epsilon > 0$, a finite set $S \in G$, and an integer $n_k > 0$ be given. We claim that there is $n_{k+1} > n_k$ such that the Følner tile $K_{n_{k+1}}$ and the set $K'_{n_{k+1}} = K_{n_k}(K_{n_{k+1}} \cap L_{n_k})$ satisfy the following condition:

$$\frac{|K_{n_{k+1}} \triangle K'_{n_{k+1}}|}{|K'_{n_{k+1}}|} < \frac{\epsilon}{3}$$

To prove the claim, note that $|K'_{n_{k+1}}| = |K_{n_k}||K_{n_{k+1}} \cap L_{n_k}| = |K_{n_k}|[L_{n_k} : L_{n_{k+1}}] = |K_{n_{k+1}}|$. We also observe that $G = K_{n_k} L_{n_k}$ is partitioned into three sets:

- G_1 = union of right translates of K_{n_k} that are subsets of $K_{n_{k+1}}$
- G_2 = union of right translates of K_{n_k} that are subsets of $K^c_{n_{k+1}}$
- G_3 = union of right translates of K_{n_k} that intersect both $K_{n_{k+1}}$ and $K^c_{n_{k+1}}$

Then, $G_1 \subset K_{n_{k+1}} \cap K'_{n_{k+1}} \subset K_{n_{k+1}} \cup K'_{n_{k+1}} \subset G_2^c$ and $G_2^c \setminus G_1 = G_3$. Consequently,

$$\begin{aligned} K_{n_{k+1}} \triangle K'_{n_{k+1}} \subset G_3 &= \bigcup \{K_{n_k} l : l \in L_{n_k} \text{ with } K_{n_k} l \cap K_{n_{k+1}} \neq \emptyset \text{ and } K_{n_k} l \cap K^c_{n_{k+1}} \neq \emptyset\} \\ &\subset K_{n_k} \partial_{K_{n_k}^{-1}} K_{n_{k+1}} = K_{n_k}(K_{n_k}^{-1} K_{n_{k+1}} \cap K_{n_k}^{-1} K^c_{n_{k+1}}) \subset \partial_{K_{n_k} K_{n_k}^{-1}} K_{n_{k+1}} \end{aligned}$$

Hence, by the previous lemma, there exists $n_{k+1} > n_k$ such that the claim is true. Enlarge n_{k+1} if necessary, so that

$$|K_{n_{k+1}} \triangle K_{n_{k+1}} s| < \frac{\epsilon}{3} |K_{n_{k+1}}| \text{ and } |K_{n_{k+1}} \triangle s K_{n_{k+1}}| < \frac{\epsilon}{3} |K_{n_{k+1}}|, \text{ for all } s \in S$$

Then we claim that $K'_{n_{k+1}}$ satisfies:

$$|K'_{n_{k+1}} \triangle K'_{n_{k+1}} s| < \epsilon |K'_{n_{k+1}}| \text{ and } |K'_{n_{k+1}} \triangle s K'_{n_{k+1}}| < \epsilon |K'_{n_{k+1}}|, \text{ for all } s \in S$$

We show the left Følner condition below; the same argument works for the right Følner condition too. First observe that

$$K'_{n_{k+1}} \triangle s K'_{n_{k+1}} \subset (K'_{n_{k+1}} \triangle K_{n_{k+1}}) \cup (K_{n_{k+1}} \triangle s K_{n_{k+1}}) \cup (s K_{n_{k+1}} \triangle s K'_{n_{k+1}})$$

to get

$$\begin{aligned} |K'_{n_{k+1}} \triangle s K'_{n_{k+1}}| &\leq |K'_{n_{k+1}} \triangle K_{n_{k+1}}| + |K_{n_{k+1}} \triangle s K_{n_{k+1}}| + |s K_{n_{k+1}} \triangle s K'_{n_{k+1}}| \\ &< \frac{\epsilon}{3} |K'_{n_{k+1}}| + \frac{\epsilon}{3} |K'_{n_{k+1}}| + \frac{\epsilon}{3} |K'_{n_{k+1}}| = \epsilon |K'_{n_{k+1}}| \text{ for all } s \in S \end{aligned}$$

Therefore, $K'_{n_{k+1}}$ is a left/right Følner tile which is tiled by K_{n_k}, and thus we can replace $K_{n_{k+1}}$ by it. □

We now discuss profinite completions of groups. Let I be a directed set. We define an *inverse system* (G_i, ϕ_{ij}) as a net of topological groups G_i, $i \in I$ and continuous homomorphisms $\phi_{ij} : G_j \to G_i$ such that ϕ_{ii} is the identity morphism, for all $i \in I$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ for all $i \leq j \leq k$. In the category of topological groups, inverse systems have always a unique limit. The *inverse limit* of the inverse system (G_i, ϕ_{ij}), denoted by $\varprojlim G_i$, is the subgroup of the product $\prod_{i \in I} G_i$ consisting of sequences $(x_i)_i$ that satisfy $\phi_{ij}(x_j) = x_i$ for all $i \leq j$.

In particular, for G a residually finite group, we fix a decreasing sequence of finite index normal subgroups L_n that separates points of G. The groups G_n in the definition of the inverse system are the finite quotients G/L_n, and the homomorphisms $\phi_{nm} : G/L_m \to G/L_n$ are given by $\phi_{nm}(xL_m) = xL_n$ for $n \leq m$. Then the *profinite completion* of G with respect to these subgroups, denoted by $\tilde{G}$, is the inverse limit of the finite quotients G/L_n, that is, the subgroup of $\prod_{n\geq 1} G/L_n$ consisting of sequences $(x_nL_n)_n$ such that $x_mL_n = \phi(x_mL_m) = x_nL_n$ whenever $n \leq m$.

We denote by π_n the canonical projections onto G/L_n, $n \geq 1$ and we formulate below some well-known properties of the profinite completion of G (cf. [13]).

Proposition 5. *The profinite completion has the following properties:*

(1) *$\tilde{G}$ is a non-empty totally disconnected compact Hausdorff group.*
(2) *The sets $\pi_n^{-1}(\{xL_n\})$, $xL_n \in G/L_n$ and $n \geq 1$, form a base of compact and open sets for the topology on $\tilde{G}$.*

Proof. The first assertion follows from standard general topology arguments. To prove the second assertion, note that the sets $\pi_n^{-1}(\{xL_n\})$ are compact and open for all $xL_n \in G/L_n$ and $n \geq 1$. Let U be open in $\tilde{G}$ and $(x_nL_n)_n \in U$. Then, there are integers $n_1 < \cdots < n_m$ and open sets $U_{n_1}, \ldots, U_{n_m}$ in the respective quotients, such that $\pi_{n_1}^{-1}(U_{n_1}) \cap \cdots \cap \pi_{n_m}^{-1}(U_{n_m}) \subset U$. In particular, $\pi_{n_1}^{-1}(\{x_{n_1}L_{n_1}\}) \cap \cdots \cap \pi_{n_m}^{-1}(\{x_{n_m}L_{n_m}\}) \subset U$. But $\pi_{n_1}^{-1}(\{x_{n_1}L_{n_1}\}) \supset \cdots \supset \pi_{n_m}^{-1}(\{x_{n_m}L_{n_m}\})$, hence the conclusion. $\square$

A useful corollary is that every compact set in $\tilde{G}$ is a finite union of the above-mentioned base sets. Also, G embeds as a dense subgroup into $\tilde{G}$, and similarly, L_n is dense in $\ker \pi_n$ for all $n \geq 1$. For ease of notation, we will denote $\ker \pi_n$ by $\tilde{L_n}$ from now on. Finally, we observe that $\pi_n^{-1}(\{xL_n\}) = x\tilde{L_n}$ for every $x \in G$ (after the identification of G as a subgroup of $\tilde{G}$).

2. Construction of the Generalized Bunce–Deddens algebras

We assume that the reader is familiar with the construction of crossed products. A good reference is [12]. In our case, the reduced crossed product $A \rtimes_{\alpha,r} G$ coincides with the full crossed product $A \rtimes_\alpha G$, since G is an amenable group.

Let $q = (q_n)_n$ be a sequence of positive integers such that q_{n+1} is divisible by q_n for all $n \geq 1$. The usual way to define the *Bunce–Deddens algebra of type* q is to consider the sequence of finite groups

$$\mathbb{Z}/q_1\mathbb{Z} \to \mathbb{Z}/q_2\mathbb{Z} \to \cdots \to \mathbb{Z}/q_n\mathbb{Z} \to \cdots$$

where each term acts on $C(\mathbb{T})$ by rotations, and take the inductive limit of the corresponding crossed products. Since $C(\mathbb{T}) \rtimes \mathbb{Z}/q_n\mathbb{Z}$ is isomorphic to the algebra of $q_n \times q_n$ matrices with entries in $C(\mathbb{T})$, we can also consider the resulting algebra as the inductive limit of such matrix algebras.

A second approach is to start with the limit of the inverse sequence

$$\mathbb{Z}/q_1\mathbb{Z} \leftarrow \mathbb{Z}/q_2\mathbb{Z} \leftarrow \cdots \leftarrow \mathbb{Z}/q_n\mathbb{Z} \leftarrow \cdots$$

which is just the profinite completion $\tilde{\mathbb{Z}}$ of $\mathbb{Z}$ with respect to these subgroups. Then, consider the integers as a subgroup of $\tilde{\mathbb{Z}}$ and define the action by addition. The crossed product $C(\tilde{\mathbb{Z}}) \rtimes \mathbb{Z}$ is again the Bunce–Deddens algebra of type q, as seen by the fact that the action by rotations of the inductive limit of $\mathbb{Z}/q_n\mathbb{Z}$'s on $\mathbb{T}$ induces the action by addition of the group $\tilde{\mathbb{T}}$ ($\cong \mathbb{Z}$) of characters of $\mathbb{T}$ on the inverse limit of the $\widetilde{\mathbb{Z}/q_n\mathbb{Z}}$ ($\cong \mathbb{Z}/q_n\mathbb{Z}$).

We now generalize this second construction in the following way: Let G be an amenable residually finite group with a sequence of nested finite index normal subgroups L_n that separates points, and consider its action α by left multiplication on its profinite completion $\tilde{G}$ with respect to these subgroups. The resulting crossed products $C(\tilde{G}) \rtimes_\alpha G$ are the *generalized Bunce–Deddens algebras.* A few of their properties are straightforward to check. Since $\tilde{G}$ is compact, $C(\tilde{G})$ is unital, and together with G being discrete, makes $C(\tilde{G}) \rtimes G$ unital as well. Separability is also evident: $\tilde{G}$ is metrizable, therefore $C(\tilde{G})$ is separable, and since G is countable, $C(\tilde{G}) \rtimes G$ is clearly separable too.

Recall that an action of G on a locally compact Hausdorff space X is *free* if every stabilizer $\{g \in G : gx = x\}$ is trivial, and *minimal* if every orbit $\{gx : g \in G\}$ is dense in X. Since G is amenable and its action on $\tilde{G}$ is free and minimal, the crossed product $C(\tilde{G}) \rtimes G$ is a simple C*-algebra, and nuclearity is a consequence of the theorem below (refer to [2] for proofs).

Theorem 6. *A C*-algebra A is nuclear, if and only if there exist contractive completely positive maps $\phi_n : A \to M_{k_n}(\mathbb{C})$ and $\psi_n : M_{k_n}(\mathbb{C}) \to A$ such that their composition approximates the identity map in the point-norm topology. In particular, if G is a discrete amenable group acting on a compact Hausdorff space X, then the crossed product $C(X) \rtimes G$ is nuclear.*

Therefore, we have concluded that:

Corollary 7. *The generalized Bunce–Deddens algebras are unital simple separable and nuclear.*

Finally, we claim that the generalized Bunce–Deddens algebras are quasidiagonal. We start with the definition of quasidiagonality.

Definition 4. A linear operator T on a separable Hilbert space $\mathcal{H}$ is *quasidiagonal* if there exists a sequence of finite rank self-adjoint orthogonal projections P_n in $\mathcal{B}(\mathcal{H})$ satisfying:

(1) $P_n \to I_{\mathcal{H}}$ as $n \to \infty$, and
(2) $\|[T, P_n]\| \to 0$ as $n \to \infty$

A separable set of operators $\mathcal{A}$ is *quasidiagonal* if every operator T in a set of dense linear span in $\mathcal{A}$ is quasidiagonal with respect to the same sequence $(P_n)_n$. An abstract C*-algebra A is *quasidiagonal* if it has a faithful representation to a quasidiagonal set of operators.

We use the following theorem from [6].

Theorem 8. *Let G be a discrete countable amenable and residually finite group with a sequence of Følner sets F_n and tilings of the form $G = K_n L_n$ with $F_n \subset K_n$ for all $n \geq 1$. Let A be a unital separable C*-algebra and let $\alpha : G \to AutA$ be a homomorphism such that*

$$\lim_{n\to\infty} \left[\max_{l \in L_n \cap K_n K_n^{-1} F_n} \|\alpha(l)a - a\| \right] = 0$$

for all $a \in A$. Assume, moreover, that A is quasidiagonal. Then $A \rtimes_\alpha G$ is also quasidiagonal.

Observe that due to Proposition 5 and the remarks after it, every function f in $C(\tilde{G})$ can be approximated within ϵ by the sum of constant functions supported on the compact open sets $x\tilde{L_n}$, where $x \in K_n$ and with n depending on ϵ and the function f. Moreover, recall that the action of any element $l \in L_n$ on $x\tilde{L_n} = \tilde{L_n}x$ leaves it invariant, since L_n is embedded in $\tilde{L_n}$ for all $n \geq 1$. Hence,

$$\alpha(l)\chi_{x\tilde{L_n}} = \chi_{x\tilde{L_n}} \text{ for any } l \in L_n$$

and by the triangle inequality,

$$\max_{l \in L_n} \|\alpha(l)f - f\| < 2\epsilon \text{ for } n \text{ large enough}$$

It follows that the action is almost periodic, as defined in the statement of Theorem 8, and thus we have the following:

Theorem 9. *Every generalized Bunce–Deddens algebra is quasidiagonal.*

3. Almost AF groupoids and further properties

In this section we introduce groupoids and focus especially on transformation groups as such. The connection with crossed products is the following: The groupoid algebra of a transformation group $X \rtimes G$ is canonically isomorphic to the crossed product $C(X) \rtimes G$.

Definition 5. A *groupoid* is a set $\mathcal{G}$ with an associative product defined on the subset of $\mathcal{G} \times \mathcal{G}$ consisting of *composable pairs*, and an inverse defined everywhere. The inverse satisfies $(x^{-1})^{-1} = x$ and every element makes a composable pair with its inverse (in either order), but also $x^{-1}x$ need not equal xx^{-1}. The pair (x, y) is composable if and only if $x^{-1}x = yy^{-1}$, in which case both $x^{-1}xy = y$ and $xyy^{-1} = x$ are true. The set of elements of the form $x^{-1}x$ for $x \in \mathcal{G}$ is the *unit space* of $\mathcal{G}$, denoted by $\mathcal{G}^0$. The element $s(x) = xx^{-1}$ is called the *source* of $x \in \mathcal{G}$, and $r(x) = x^{-1}x$ is its *range*.

We focus on $\mathcal{G} = \tilde{G} \rtimes G$ from now on. The action is by left multiplication, and the elements of $\tilde{G} \rtimes G$ are of the form (x, g), with $x \in \tilde{G}$ and $g \in G$. The product is defined on pairs $((x, g), (y, h))$ such that $y = gx$, by the formula $(x, g)(y, h) = (x, hg)$, and the inverse $(x, g)^{-1} = (gx, g^{-1})$. $\tilde{G} \rtimes G$ is thus a groupoid, with unit space isomorphic to $\tilde{G}$. Think of (x, g) as an arrow from $x \in \tilde{G}$ to $gx \in \tilde{G}$. The fact that all arrows are defined uniquely by their endpoints is a consequence of the action being free, and makes $\tilde{G} \rtimes G$ a *principal* groupoid. Also, it is a *Cantor* groupoid, which is defined as a second countable locally compact Hausdorff etale groupoid, whose unit space is the Cantor set, equipped with a Haar system of counting measures. Indeed, for transformation groups $X \rtimes G$ it is sufficient that X is homeomorphic to the Cantor set and that G is discrete and countable (cf. [7]). Moreover, an open subgroupoid of a Cantor groupoid with the same unit space is itself a Cantor groupoid.

We continue with the definion of an AF groupoid from [4]

Definition 6 (Giordano–Putnam–Skau)**.** A Cantor groupoid $\mathcal{G}$ is called *approximately finite* (AF) if it is the increasing union of a sequence of compact open principal Cantor subgroupoids, with each of them containing $\mathcal{G}^0$.

We show that $\mathcal{G} = \tilde{G} \rtimes G$ contains an AF groupoid, by constructing a nested sequence of compact open subgroupoids. Based on the tiling $G = K_n L_n$ of the group G, and the fact that G acts freely on $\tilde{G}$, we get a tiling

$$\tilde{G} = \bigcup_{xL_n \in G/L_n} \pi_n^{-1}(\{xL_n\}) = K_n \tilde{L_n}$$

of its profinite completion. We then define the following subsets of $\mathcal{G}$:

$$\mathcal{G}_n = \{(gx, g_1 g^{-1}) : x \in \tilde{L_n},\, g, g_1 \in K_n\}$$

It is easy to check that $\mathcal{G}_n$ are subgroupoids of $\mathcal{G}$ with the same unit space. They are compact and open in $\mathcal{G}$ as a consequence of $\tilde{L_n}$ being such. In addition, Theorem 4 allows us to obtain a subsequence of these subgroupoids which is nested, i.e., to find integers $n_1 < n_2 < \cdots$ such that $\mathcal{G}_{n_k} \subset \mathcal{G}_{n_{k+1}}$ for all $k \geq 1$. Indeed, if $(gx, g_1 g^{-1}) \in \mathcal{G}_{n_k}$, then write $gx = hy \in K_{n_{k+1}} \tilde{L_{n_{k+1}}}$. Since $K_{n_{k+1}}$ has a partition of the form $\bigcup_{l \in L} K_{n_k} l$ for a suitable finite set $L \subset L_{n_k}$, we have $K_{n_{k+1}} y = \bigcup_{l \in L} K_{n_k} l y$, hence $gx \in K_{n_k} l_1 y$ for some $l_1 \in L$, which gives $x = l_1 y$ (since the action is free). Then, $h_1 = g_1 l_1 \in K_{n_{k+1}}$ and thus $(gx, g_1 g^{-1}) = (hy, h_1 h^{-1}) \in \mathcal{G}_{n_{k+1}}$.

The groupoid algebra of an AF groupoid is an AF algebra, and conversely, to any AF algebra we can associate a unique AF groupoid with groupoid algebra the AF algebra we started with (cf. [9]). Observe however that for elements of infinite order in a group G, the corresponding elements in $C^*(G)$ cannot have finite spectrum. Therefore, $\mathcal{G} = X \rtimes G$ is not an AF groupoid, as long as G is not a locally finite group. Yet, it may happen that $\mathcal{G}$ is not 'much bigger' than an open AF subgroupoid, in a sense that N. C. Phillips made precise in [7]. The following definition applies when the groupoid algebra of $\mathcal{G}$ is simple. A *graph* is a subset of $\mathcal{G}$ for which the restrictions of the source and range maps are injective.

Definition 7 (Phillips)**.** Assume $\mathcal{G}$ is a Cantor groupoid such that $C_r^*(\mathcal{G})$ is simple. Then $\mathcal{G}$ is *almost AF* if it contains an open subgroupoid $\mathcal{G}_{AF}$ with the same unit space and the following property: for every compact subset C of $\mathcal{G} \setminus \mathcal{G}_{AF}$ and every $m \geq 1$, there exist compact graphs $C_1, \ldots, C_m$ in $\mathcal{G}_{AF}$ with source $s(C_i) = s(C)$ for $i = 1, \ldots, m$ and disjoint ranges.

The crucial step is to show that $\mathcal{G} = \tilde{G} \rtimes G$ is an almost AF groupoid. A more general result of this sort appears in [7]; however Phillips has to assume finite generation (which we do not need).

Theorem 10. *The groupoid $\mathcal{G} = \tilde{G} \rtimes G$ is almost AF for every discrete countable amenable residually finite group G and every profinite completion $\tilde{G}$ associated to a separating nested sequence of finite index normal subgroups L_n of G.*

Proof. We are inspired from the proof of Theorem 6.9 in [7]. We will use the already established notation. We will also identify the unit space of $\mathcal{G}$ with $\tilde{G}$ to simplify notation. Set $\mathcal{G}_{AF} = \bigcup_{k \geq 1} \mathcal{G}_{n_k}$ which is open in $\mathcal{G}$. The goal is to verify the last condition of Definition 7. To that end, consider a compact set $C \subset \mathcal{G} \setminus \mathcal{G}_{AF}$ and an integer $m \geq 1$. Define

$$S = \{g \in G : (\tilde{G} \times \{g\}) \cap C \neq \emptyset\}$$

which is a finite set because C is compact.

By Theorem 4, there exists $n \in \{n_1, n_2, \dots\}$ so that $|K_n \triangle sK_n| < \dfrac{1}{|S|m}|K_n|$ for all $s \in S$. For $(y,s) \in C$, $y \in \tilde{G} = K_n\tilde{L_n}$, so $y = gx$ for some $g \in K_n$ and $x \in \tilde{L_n}$. However, $(y,s) \notin \mathcal{G}_n$, hence $(y,s) \neq (gx, g_1g^{-1})$ for any $g_1 \in K_n$. It follows that $g \in K_n \setminus s^{-1}K_n$. If $K = \displaystyle\bigcup_{s\in S}(K_n \setminus s^{-1}K_n)$, then we have $s(C) \subset \displaystyle\bigcup_{g\in K} g\tilde{L_n}$ with

$$|K| \leq \sum_{s\in S}|K_n \setminus s^{-1}K_n| \leq |S||K_n \triangle sK_n| < \frac{1}{m}|K_n|$$

Therefore, there exist m injective functions $\sigma_1, \dots, \sigma_m : K \to K_n$ with disjoint ranges. As a result, the compact sets $C_i = \displaystyle\bigcup_{g\in K}[(s(C) \cap g\tilde{L_n}) \times \{\sigma_i(g)g^{-1}\}]$, $i = 1, \dots, m$, satisfy

$$s(C_i) = \bigcup_{g\in K}(s(C) \cap g\tilde{L_n}) = s(C)$$

and their ranges

$$r(C_i) = \bigcup_{g\in K}[\sigma_i(g)g^{-1}s(C) \cap \sigma_i(g)\tilde{L_n}] \subset \bigcup_{g\in K}\sigma_i(g)\tilde{L_n}$$

are disjoint by definition of the functions σ_i.

Moreover, if $(gx, \sigma_i(g)g^{-1})$ and $(hy, \sigma_i(h)h^{-1})$ are two elements of C_i, then

$$s(gx, \sigma_i(g)g^{-1}) = gx = hy = s(hy, \sigma_i(h)h^{-1}) \text{ forces } g = h \text{ and } x = y$$

and the same is true for the ranges. Hence C_i is a graph for all $i = 1, \dots, m$. Finally, $C_i \subset \mathcal{G}_n$ because the functions σ_i map into K_n for every $i = 1, \dots, m$. □

The significance of almost AF groupoids comes from Phillips's theorem below. But first, let us recall the following notions:

Definition 8. A (unital) C*-algebra has *real rank zero* if the self-adjoint invertible elements are dense in the set of all self-adjoints (Brown–Pedersen, [1]). A C*-algebra has *stable rank one* if the invertible elements are dense in the algebra (Rieffel, [10]).

Theorem 11 (Phillips)**.** *Let $\mathcal{G}$ be an almost AF groupoid, and assume $C_r^*(\mathcal{G})$ is simple. Then $C_r^*(\mathcal{G})$ has real rank zero and stable rank one. Moreover, $C_r^*(\mathcal{G})$ has comparability of projections: for p, q projections in $\mathcal{M}_\infty(C_r^*(\mathcal{G}))$ with $\tau(p) < \tau(q)$ for all normalized traces τ on $C_r^*(\mathcal{G})$, p is Murray–von Neumann equivalent to a subprojection of q. Finally, there is a bijection between normalized traces on $C_r^*(\mathcal{G})$ and invariant Borel probability measures on $\mathcal{G}^0$.*

A Borel measure μ on $\mathcal{G}^0$ is *invariant* if for every $f \in C_c(\mathcal{G})$, the following is true:

$$\int_{\mathcal{G}^0}\left(\sum_{g\in\mathcal{G}:s(g)=x} f(g)\right)d\mu(x) = \int_{\mathcal{G}^0}\left(\sum_{g\in\mathcal{G}:r(g)=x} f(g)\right)d\mu(x)$$

The correspondence between traces and invariant measures on the unit space is obtained for more general groupoids and is more explicit in [7].

We are now able to prove:

Theorem 12. *Every generalized Bunce–Deddens algebra has real rank zero, stable rank one, comparability of projections, and a unique trace.*

Proof. We combine Theorems 10 and 11 to get real rank zero, stable rank one and comparability of projections. The unique trace is obtained as follows: Observe that

$$S = \{g \in G : (\tilde{G} \times \{g\}) \cap \operatorname{supp} f \neq \emptyset\}$$

is finite, and hence, that the integrals in the equation that gives the invariance of a Borel measure on $\mathcal{G}^0$ are:

$$\sum_{s \in S} \int_{\tilde{G}} f(x, s) d\mu(x), \text{ and } \sum_{s \in S} \int_{\tilde{G}} f(s^{-1}x, s) d\mu(x)$$

We now see that the two integrals are equal if and only if the measure μ is G-invariant. However, we can show that the normalized Haar measure is the only G-invariant probability measure on $\tilde{G}$. Indeed, if μ is any G-invariant measure, $f \in C_c(\tilde{G})$, and $(y_n)_n$ is a sequence of elements of G converging to $y \in \tilde{G}$, then

$$\int_{\tilde{G}} f(x) d\mu(x) = \int_{\tilde{G}} f(y_n^{-1}x) d\mu(x) \longrightarrow \int_{\tilde{G}} f(y^{-1}x) d\mu(x), \text{ as } n \to \infty$$

Therefore μ is also $\tilde{G}$-invariant, and we are done. □

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Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH, 45221

E-mail address: stefanos.orfanos@uc.edu